Possibility theorems for social welfare functions

Storcken, Antonius Jozef Agnes

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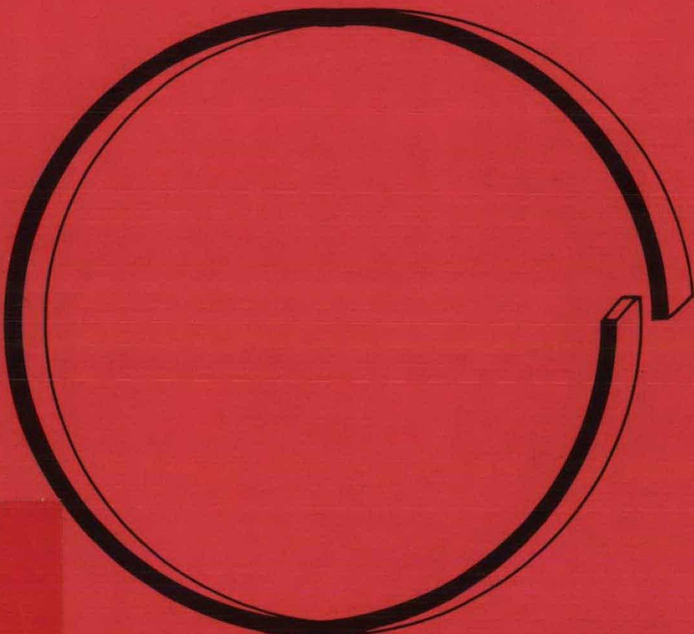

Ton Storcken

Possibility Theorems for Social Welfare Functions

PROEFSCHRIFT

ter verkrijging van de graad van doctor aan de
Katholieke Universiteit Brabant,
op gezag van de rector magnificus,
prof. dr. R.A. de Moor,
in het openbaar te verdedigen
ten overstaan van een door het college
van dekanen aangewezen commissie
in de aula van de Universiteit
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door

Antonius Jozef Agnes Storcken

geboren te Geleen

Promotores : Prof. dr. P.H.M. Ruys
Prof. dr. S.H. Tijs
Prof. dr. H.C.M. de Swart

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CONTENTS:

STELLINGEN

behorend bij het proefschrift "Possibility Theorems for Social Welfare Functions" van A.J.A. Storcken.

I. Laat $a_0, a_1, a_2, \ldots$ een rij rangtelwoorden zijn uit een taal zeg T. Noteer het aantal letters van zo'n woord a_i met $|a_i|$. Definieer $\lim_{i \to \infty} a_i = \langle b_0, b_1, \ldots, b_t \rangle$, d.e.s.d.a. er een index k is, zodanig dat voor alle $i \geq k$ $a_i = b_j$, waarbij $j = i-k$ (modulo t+1). Veronderstel dat voor elke index $i \geq 1$ de rekenkundige waarde van a_{i+1} gelijk is aan $|a_i|$. Dan

(a) als T de nederlandse, duitse, resp. engelse taal is, dan is $\lim_{i \to \infty} a_i$ gelijk aan <vier>, <vier>, resp. <four>,

(b) als T de franse taal is, dan is $\lim_{i \to \infty} a_i$ = <cinq, quatre, six, trois>.
(We identificeren <cinq, quatre, six, trois> met b.v. <quatre, six, trois, cinq>.)

II Laat $a_0, a_1, a_2, \ldots$ een rij rangtelwoorden uit de nederlandse taal zijn, zodanig dat voor elke index $i \geq 1$ de rekenkundige waarde van a_{i+1} gelijk is aan $|a_i|^2$. Dan is $\lim_{i \to \infty} a_i$ òf gelijk aan
<vijfhonderdnegenentwintig,zeshonderdvijfentwintig>
òf gelijk aan <vijfhonderdzesenzeventig>.

III Er is geen koppelprocedure, die zowel stabiel als consistent is.
Laat V en M twee disjuncte verzamelingen zijn met $|V| = |M| \geq 3$. Laat $L_m(A)$ het m-voudige cartesische product zijn van de verzameling van lineaire ordeningen, L(A), op de verzameling A. Definieer

$P(V,M) := \{\langle r_A, r_B\rangle$: Er zijn m, $A \subseteq V$, en $B \subseteq M$, zodanig dat $|A| = |B| = m$, $r_A \in L_m(A)$ en $r_B \in L_m(B)\}$ en

$\Sigma(A,B)$ is de verzameling van bijecties van A naar B.

Een <u>koppelprocedure</u> K is nu een afbeelding welke aan alle $\langle r_A, r_B\rangle \in P(V,M)$ een bijectie $K(r_A, r_B) \in \Sigma(A,B)$ toevoegt.

Een koppelprocedure K is <u>stabiel</u>, d.e.s.d.a. voor alle $\langle r_A, r_B\rangle \in P(V,M)$ er geen $i,j \in B$ en $i',j' \in A$ zijn zodanig dat $K(r_A,r_B)(i) = i'$, $K(r_A,r_B)(j) = j'$, $j' > i' : R^i$ en $i > j : R^{j'}$.

Een koppelprocedure K is <u>consistent</u>, d.e.s.d.a. voor alle $\langle r_A, r_B\rangle \in P(V,M)$ en voor alle $j \in A$ en alle $i \in B$, met $K(r_A,r_B)(j) = i$: $K(r_A|_{A-\{j\}}, r_B|_{B-\{i\}}) = K(r_A,r_B)|_{A-\{j\}}$.

(Zie Gale D. & Shapley L.S., 1962, College admissions and the stability of marriage, American Mathematical Monthly, 69,p. 9 - 15, en Roth A.E., 1982, The economics of matching: stability and incentives, Mathematics of Operations Research, 7, p. 617 - 628)

IV Laat $V \subseteq Å$ een classificeerbare vezameling van ordeningen zijn en H een "decision procedure" op $\alpha(V)$ (Zie def. 4.2.4). Dan zijn IV.1 en IV.2 equivalent.

IV.1 H is sterk positief associatief, dat is $\{x\} \cap H(R_X^2) \subseteq H(R_X^1)$ voor alle $x \in X$ en $R_X^1, R_X^2 \in V$ met $\langle R_X^1, R_X^2\rangle \in \Omega(\{x\} \times X)$ (Zie def. 1.4.4).

IV.2 $V \subseteq A(U)$ (de verzameling van sterk complete acyclische ordeningen) en voor alle $\langle X, R_Y\rangle \in \alpha(V)$: $H(X,R_Y) = \mathrm{Best}(R_Y|_X)$ (:= $\{x \in X : x \geq y : R_Y$ voor alle $y \in X\}$).

Dus "rationalizeerbaarheid" van keuzegedrag komt overeen met een monotonie eigenschap voor keuzegedrag.

V Zij C een keuzefunctie van $2^A - \{\varphi\}$ naar A, zodanig dat A eindig is en $C(X) \varepsilon X$ voor alle $X \varepsilon 2^A - \{\varphi\}$.
C is reconstrueerbaar met ordeningen (Zie def. 4.2.4), d.e.s.d.a. er een lineaire ordening $R_A \varepsilon L(A)$ is zodanig dat voor alle $X \varepsilon 2^A - \{\varphi\} : \{C(X)\} = \text{Best}(R_{A|_X})$.
Aangezien reconstrueerbaarheid een zwakkere eis is dan "rationalizeerbaarheid" en lineaire ordeningen de meest gestructureerde ordeningen zijn en derhalve tot de simpelste structuren leiden, volgt dat "rationalizeerbaar" keuzegedrag, welk bovendien tot enkelvoudige keuzes leidt, simpel is.

VI Er is een sociaal keuze theoretisch bewijs voor de volgende stelling:
Zij R een zwakke ordening van $\mathbb{R}^n$, zodanig dat voor alle $x,y,z \varepsilon \mathbb{R}^n$, met $x \neq y$:

VI.1 als voor alle $i \varepsilon \{1,2,\dots,n\}$ $x_i \geq y_i$, dan $x > y : R$ (meer is beter)

VI.2 als $x > y : R$, dan $x + z > y + z : R$ (translatie invariantie).

VI.3 als voor alle $i \varepsilon \{1,2,\dots,n\}$ $x_i > z_i > y_i$ of $y_i > z_i > x_i$, dan $x > z > y : R$ of $y > z > x : R$ (tussen relatie behoudend).

Dan is er een rangschikking i1 i2 i3 ... in van $\{1,2,\dots,n\}$ zodanig dat voor alle $x,y \varepsilon \mathbb{R}^n$: $x > y : R$, d.e.s.d.a. er is een nummer k met $x_{it} = y_{it}$ voor alle $t < k$ en $x_{ik} > y_{ik}$ (R is een lexicografische ordening).

VII Er is een sociaal keuze theoretisch bewijs voor de volgende stelling:

Zij R een zwakke ordening van $\mathbb{R}^n$, zodanig dat voor alle $x,y,z \in \mathbb{R}^n$, met $x \neq y$, en $\alpha > 0$:

VI.1 als voor alle $i \in \{1,2,...,n\}$ $x_i \geq y_i$, dan $x > y : R$ (meer is beter)

VI.2 als $x > y : R$, dan $x + z > y + z : R$ en $\alpha \cdot x > \alpha \cdot y : R$.

VI.3 als σ een permutatie is van $\{1,2,...,n\}$ en $\sigma x := \langle x_{\sigma(1)},...,x_{\sigma(n)} \rangle$, dan als $x > y : R$, dan $\sigma x > \sigma y : R$ (neutraliteit).

Dan $x \geq y : R$, d.e.s.d.a. $\sum_{i \in \{1,...,n\}} x_i \geq \sum_{i \in \{1,...,n\}} y_i$, voor alle $x, y \in \mathbb{R}^n$.

Deze en de vorige stelling geeft een karakterisering van ordeningen van bijvoorbeeld goederen bundels of verdelingen van inkomens, op basis van de Sociale Keuze Theorie.

VIII Voor elke $\sigma \in [0,1]$ is er een "inequality index" die voldoet aan de eisen "normalization", "Schur-convexity", "intermediate inequality concept" op basis van σ, "population replication principle" en "extensibility". (Zie Bossert W., A note on intermediate inequality indices which are quasilinear means, Institut für Wirtschaftstheorie und Operations Research, Universität Karlsruhe, discussion paper 289).

CHAPTER 1 SOCIAL CHOICE PROBLEMS

§ 1.1 Introduction to basic notions

We are all familiar with situations in which a group of individuals has to agree upon a collective choice or preference, e.g., the election of a president or chairman, the ranking of true-bred dogs by a jury at a dogshow, or the amendment of a bill in parliament. Although much time and consideration is absorbed by discussions and individual points of view about the possible collective standpoints, the methods which yield the collective agreement are seldom at stake. Therefore, it is surprising that the formalizations of these methods are contaminated by serious defects, such as manipulation of the outcome or non-monotonic behaviour between two possible outcomes.

In fact, due to the surprising failure to construct defect-free methods by which a group of individuals may determine its collective decision, a formal theory of collective decision making has been developed. This theory is called social choice theory. Since the theory of social choice is a formal theory with rather weak assumptions, it may deal with several political, economic and ethical problems. Therefore, it is not astonishing that this theory is formalized in several ways. In this monograph social choice is formalized by means of a mathematical framework and the emphasis lays on the mathematics induced by social choice problems.

In this chapter some basic notions, assumptions and notations are introduced. Furthermore, several of the defects of some collective decision procedures by which a group of individuals determines its collective decision, are discussed. Finally, some prelimimary work is done.

The present work is far from being a review or handbook of social choice theory. In, for instance, Sen [1970], Sen [1986], Arrow [1978], Black [1987], Moulin [1983], Arrow & Raymond [1986], Peleg [1984], Pattanaik [1978], Schofield [1985], Fishburn [1973], Farquharson [1969] and Brams & Fishburn [1982], one finds reviews, other formalizations of social choice problems, other (here left unspoken) defects and other mathematical formalizations of social choice theory.

This work concentrates on constitutional decision making of a group of individuals about a number of alternatives. The notions of individuals as well as alternatives appear in all problems of social choice theory. They will be regarded as primitive notions. Furthermore, hereafter it is assumed that both the number of individuals and the number of alternatives are finite, which makes it possible to formalize these basic notions as sets.

Definition 1.1.1 Society

A Society Γ is an ordered pair $\langle A,N \rangle$ of non-empty and finite sets, where A is the set of alternatives and N is the set of individuals.

■

Clearly, the non-emptiness excludes odd posibilities, e.g., an empty set of individuals which decides between the alternatives in A. Note that by definition 1.1.1 trivial societies, i.e., where the set of individuals or the set of alternatives is a singleton, are admittable.

In literature societies with an infinite set of alternatives or individuals have been studied, see e.g., Kirman & Sondermann [1972] and Chichilnisky & Heal [1983]. The greatest part of literature of social choice theory, however, is concerned with finite sets of alternatives as well as of individuals. As mentioned above, we retain in this monograph the tradition of the finiteness of the sets of individuals and alternatives. A disadvantage of this approach is that a lot of fruitful mathematics, such as measure theory and analysis, are not applicable here.

Notation 1.1.2 Basic notations of societies

Usually, a,b,c,d,e,f,x,y,z, or $a_1,a_2,\ldots,a_p$ are variables over the set of alternatives A, and i,j,k,l,n and m are variables over the set of individuals N.

If N has n elements, we interpret N as the set $\{1,2,3\ldots,n\}$. Most frequently it is supposed that N has n elements and A has p elements.

■

It is expected that the reader is familiar with the following standards of set theory.

Notation 1.1.3 Basic notations concerning set theory

Below there is a list of two columns. In the left a relation on sets or an operator is mentioned, in the right its notation.

cardinality	$\|\ldots\|$
intersection	$\cap$
union	$\cup$
symmetric difference	$\triangle$
cartesian product	X
relative complement	-
element of	ε
contained in	$\subseteq$
strictly contained in	$\subsetneq$ or $\subset$
not element of	$\notin$
not contained in	$\nsubseteq$
the empty set	φ
is equal to	=
is not equal to	$\neq$
power set	$2^{\cdots}$

■

We assume that social choice or social preferences depend only on the profile of individual preferences and on the constitutional rule of aggregation. Hence, collective group decisions, choices or preferences will be formalized in terms of decision or aggregation mechanisms on profiles of individual preferences. In fact, these mechanisms turn out to be special functions, i.e., welfare functions and choice correspondences while preferences are special relations on the set of alternatives, such as linear orderings, weak orderings, quasi-orderings, semi-orderings, partial orderings or interval orderings.

Usually, in social choice theory as well as in other theories, these orderings or, better, types of orderings are basic notions. Although in this chapter this is also assumed in order to simplify the problems discussed in this chapter, in chapter 2 a theory of orderings is developed. By this theory the types of orderings are no longer basic notions. Instead of these notions one other notion, named classified set of orderings, is basic. By this the set of, e.g., linear orderings becomes a derived notion. In this chapter sometimes the notion of set of orderings is used. At this stage the reader may interpret it as, e.g., the set of linear orderings or the set of weak orderings or the set of quasi-orderings, and so on. By chapter 2 this notion set of orderings becomes much more rich than we are able to explain here. To indicate the richness the reader is invited to cast a glance at the inclusion diagram on page ... which shows several classified sets of orderings discussed in the following chapter.

1.1.4 Notations, Definitions and Interpretations of orderings

A *binary relation* R on A is a subset of the cartesian product A x A. Hence $R \subseteq A \times A$.

R is *reflexive*, iff for all $x \in A$: $\langle x,x\rangle \in R$.

R is *antisymmetric*, iff for all $\langle x,y\rangle \in A \times A$:
if $\langle x,y\rangle \in R$ and $\langle y,x\rangle \in R$, then $x = y$.

R is *complete*, iff for all $\langle x,y\rangle \in A \times A$, with $x \neq y$:
$\langle x,y\rangle \in R$ or $\langle y,x\rangle \in R$.

R is *transitive*, iff for all $\langle x,y\rangle, \langle y,z\rangle \in R$: $\langle x,z\rangle \in R$.

$\bar{a}R$ indicates the *asymmetric part* of relation R and $\bar{s}R$ its *symmetric part*.

We have the following standard interpretations.

If $\langle x,y\rangle \in R$, then x is *as least as good as* y (according to R). Notation: $x \geq y : R$.

If $\langle x,y\rangle \in R$ and $\langle y,x\rangle \notin R$, then x is *prefered to* y (according to R). Notation: xy : R or x.——→——.y or $x > y : R$.

If $\langle x,y\rangle \in R$ and $\langle y,x\rangle \in R$, then x is *indifferent to* y (according to R). Notation: (xy) : R or x.———.y or $x \sim y : R$.

If $\langle x,y\rangle \notin R$ and $\langle y,x\rangle \notin R$, then x is *incomparable to* y (according to R). Notation: $\binom{x}{y}$: R or x.   .y or x.------.y.

The reader might think that notations like $x \geq y : R$ are contaminated with a lot of redundant information. Although this thought seems to be true at this stage, this redundancy disappears later on when, e.g., relations on relations are introduced.

Often a compound notation is used, such as $\binom{(xy)}{xz} : R$ or $x(yz) : R$. These notations should be interpreted as unions of the pair notations. Hence, $x(yz) : R$ is the abbreviation of $xy : R$ together with $xz : R$ and $(yz) : R$. $\binom{(xy)}{xz} : R$ is the abbreviation of $(xy) : R$, together with $xz : R$, $\binom{x}{x} : R$, $\binom{y}{x} : R$, $\binom{y}{z} : R$ and $\binom{x}{z} : R$. Note that $\binom{x}{x} : R$, $\binom{y}{x} : R$ and $\binom{x}{z} : R$ are meaningless here. Hence, $\binom{(xy)}{xz} : R$ is the abbreviation of $(xy) : R$, together with $xz : R$ and $\binom{y}{z} : R$. Note that $\binom{xy}{yx} : R$ and $(xy) : R$ reveal the same information of R in relation to x and y. Hence, $xy : R$ logically does not imply: $\langle x,y \rangle \in R$ and $\langle y,x \rangle \notin R$. But as a notational convention from now on it is assumed that R is completely described by its notation. Hence, $xy : R$ means $R = \{\langle x,x \rangle, \langle y,y \rangle, \langle x,y \rangle\}$ and $R \subseteq \{x,y\} \times \{x,y\}$.

We define two sets of orderings on A:

the set of <u>linear orderings</u> $L(A)$ on A:

$L(A) := \{R \subseteq A \times A : R$ is reflexive, complete, antisymmetric and transitive$\}$,

the set of <u>weak orderings</u> $W(A)$ on A:

$W(A) := \{R \subseteq A \times A : R$ is reflexive, complete and transitive$\}$.

It is evident that:

if $|A| = 1$, then $L(A) = W(A)$, and

if $|A| \geq 2$, then $L(A) \subsetneq W(A)$.

■

The notations introduced above will be explained in the following example.

Example 1.1.5

Let $\begin{bmatrix}(xy)\\(xz)\\yz\end{bmatrix}$: R. R is a relation on {x,y,z}.

R = {<a,b> ε A x A: x ε {a,b}, a = b, or <y,z> = <a,b>}. It has the following graphical representation: y.——.x——.z

Since <z,x>,<x,y> ε R and <z,y> ∉ R, it follows that R is not transitive.

Furthermore, since a linear ordering orders the alternatives from better to worse, we denote these orderings often by their better to worse sequence. If R ε L(A) and $A = \{a_1,a_2,...,a_p\}$ and a_1 is preferred to a_2, a_2 is preferred to a_3 and so on, then R is often denoted as $a_1a_2a_3...a_p$: R. (Note that this coincides with the notation introduced earlier).

If R is a weak ordering, its symmetric part forms an equivalence relation. The equivalence classes induced by this relation are often refered to by <u>indifference classes</u>. Now, if R ε W(A) its indifference classes can be ordered in a linear way. We get relations of the following form:

$(a_1a_2a_3...a_{s_1})(a_{s_1+1}...a_{s_2})(a_{s_2+1}...a_{s_3})...(a_{s_t+1}...a_p)$:R'.

In R' $\{a_1a_2a_3...a_{s_1}\},\{a_{s_1+1}...a_{s_2}\}...\{a_{s_t+1}...a_p\}$ are the indifference classes.

■

Next, the dependency of the individual preferences on the decision procedures can be formalized. Let Γ = <A,N> be a society with N = {1,2,...,n}. These numbers are just identifiers. Let us denote the preference relation of individual 1 by R^1, that of 2 by R^2 and so on. A profile or combination of individual preference relations, $<R^1,R^2,...,R^n>$, is denoted by r. Obviously $r \in \underbrace{L(A) \times L(A) \times ... \times L(A)}_{\text{n-times.}} =: L_n(A)$ the n-fold cartesian product of L(A), if all the individual preferences $R^1,R^2,...,R^n$ are linear orderings. $L_n(A)$ can be interpreted as the set of possible profiles, where all the individual preferences are linear orderings. Similarly $W_n(A)$ is the set of all possible profiles, where all the individual preferences are weak orderings.

Now, for any given combination of individual preferences, r, the constitution or decision rule is assumed to yield a decision. This can either be a unique ordering on A, in which case we speak of an aggregated or a social preference of the society, or a choice of alternatives, i.e., a non-empty subset of A, in which case we speak of a group choice.

Definition 1.1.6 Welfare function Choice Correspondence

Let $\Gamma = \langle A,N \rangle$ be a society, V(A) and U(A) sets of orderings on A and $V_n(A)$ the n-fold cartesian product of V(A), where $n = |N|$.

A function F from $V_n(A)$ to U(A) is called a welfare function on Γ (from $V_n(A)$ to U(A)).

A function C from $V_n(A)$ to $2^A - \{\varphi\}$ is called a choice correspondence on Γ (from $V_n(A)$ to $2^A - \{\varphi\}$).

■

A welfare function F is a decision procedure, which assings to every possible combination of individual orderings, $r \in V_n(A)$, a collective ordering F(r) in U(A). F(r) is often refered to by social ordering or the ordering of the society (at profile r). Similarly a choice correspondence C yields a collective or social choice of the society C(r) at profile $r \in V_n(A)$.

The notions of welfare function and choice correspondence are too weak to describe constitutional decision procedures on their own. For this reason it is necessary to impose some conditions on these functions. In sections 1.2 up to 1.5 several conditions, often imposed on these functions in social choice theory, are discussed.

In section 1.6 it is shown that the introduced conditions can be translated, at least partly, into one language. This enables a comparison between these conditions. Section 1.7 previews the following three chapters.

§ 1.2 Score rules

A frequently used type of correpondence is that in which every individual votes for one alternative (candidate). The alternatives, which receive the greatest number of votes, are chosen. This voting rule is called relative majority voting. Consider the following example.

Example 1.2.1 Borda's criticism on voting

Let $\Gamma = \langle\{a,b,c,d,e\},\{1,2,...,75\}\rangle$ be a society.
Suppose the relative majority voting rule has to be applied to profile $r \in L_{75}(\{a,b,c,d,e\})$ as follows:

abcde : R^i for $i \in \{1,2,...,16\}$ (16 individuals),
dbcea : R^i for $i \in \{17,....,31\}$ (15 individuals),
cbeda : R^i for $i \in \{32,....,46\}$ (15 individuals),
bceda : R^i for $i \in \{47,....,61\}$ (15 individuals) and
ebcda : R^i for $i \in \{62,....,75\}$ (14 individuals).

Hence, a gets 16, b, c and d get 15 and e gets 14 votes. a is then chosen. On the other hand, if we consider the position of a in all the orderings, it is clear that a is unacceptable for nearly 80% of the society. Hence, chosing a at this profile of individual orderings will probably `evoke lots of frustation'.
This criticism on voting has already been pointed out by de Borda in 1781. (See e.g. Black [1987] and Borda [1781]). He showed that this defect could happen, because only the top positions of the individual orderings affect the outcomes of a voting procedure. He therefore introduced a procedure that takes all the positions of the alternatives in all the individual orderings into account.
This procedure, known as the Borda rule, goes as follows in this society: every individual gives 5 points to his best ordered alternative, 4 to his next best, and so on. So 1 point is given to a worst ordered alternative. The alternatives with the greatest number of points are chosen. In our example:

a gets (16x5) + (15x1) + (15x1) + (15x1) + (14x1) = 139 points
b gets (16x4) + (15x4) + (15x4) + (15x5) + (14x4) = 315 points
c gets (16x3) + (15x3) + (15x5) + (15x4) + (14x3) = 270 points
d gets (16x2) + (15x5) + (15x2) + (15x2) + (14x2) = 195 points
e gets (16x1) + (15x2) + (15x3) + (15x3) + (14x5) = 206 points

When the Borda rule is applied to the above profile r, then b is the group choice and bceda : Borda(r) the aggregated preference ordering.

■

Of course the Borda rule does not have the disadvantages of the majority rule. But a lot of information is needed to calculate the Borda score (i.e., number of points) of an alternative. At least two questions arise.

(1) Is it possible to determine a decision in a more regular way, e.g., by pairwise comparison of all alternatives.

(2) Does a small disturbance lead to a small change in the outcome of a Borda rule.

The first question is discussed in the following section. The second question is illustrated by the following example.

<u>Example 1.2.2</u> <u>Influence of disturbance</u>

Let $\Gamma = \langle\{a,b,c,d,e\},\{1,2,3,4\}\rangle$ be a society. Observe the following two profiles r and $\hat{r}$ in $W_4(\{a,b,c,d,e\})$, such that:

r :	(ea) c (bd) : R^1,	$\hat{r}$:	(ea) c (bd) : $\hat{R}^1$,
	(ea) c bd : R^2,		(ea) c db : $\hat{R}^2$,
	bd c ae : R^3,		bd c ea : $\hat{R}^3$,
	db c ae : R^4,		db c ea : $\hat{R}^4$.

Note that R^2 and $\hat{R}^2$ only differ in the ordering on their worst pairs. Only one twist of the positions of alternatives is needed to obtain R^2 from $\hat{R}^2$ or vice versa. The same is true for R^3 and $\hat{R}^3$, and for R^4 and $\hat{R}^4$. All these relations differ on precisely one pair. Let us agree that in an indifference class the points scored by the Borda rule are divided equally among the members of such an indifference class. Hence, e gets $4\frac{1}{2} = \frac{1}{2} \cdot (5 + 4)$ points from R^1. Then a gets 13 points, b $12\frac{1}{2}$, c 12, d $11\frac{1}{2}$ and e gets 11 points from r, where a gets 11, b $11\frac{1}{2}$, c 12, d $12\frac{1}{2}$ and e gets 13 points from $\hat{r}$.

Hence, abcde : Borda(r) and

edcba : Borda($\hat{r}$).

We come to the conclusion that the numbers of pairs in which Borda(r) and Borda($\hat{r}$) differ, viz. $\binom{5}{2} = 10$ pairs, is greater than the total number of pairs in which the corresponding components of r and $\hat{r}$ differ, viz. 3. Hence, a small change in the combination of individual orderings may cause a large change in the Borda outcome. The Borda rule is not <u>non-expansive</u> with respect to those changes. We do not formalize this non-expansiveness criterion here. This will be done in chapters 3 and 4.

■

The second question in the previous example has got a clear answer: No. This criticism on the Borda rule is not well known. Another defect of this Borda rule has frequently been discussed in literature.

<u>Example 1.2.3</u> <u>Manipulation</u>

Let $\Gamma = \langle\{a,b,c,d,e\},\{1,2,\ldots,75\}\rangle$ be a society and let r and $\hat{r}$ be two profiles in $L_{75}(\{a,b,c,d,e\})$, such that:

r : abecd : R^1,
aebcd : R^i for $i \in \{2,\ldots,35\}$,
eabcd : R^i for $i \in \{36,\ldots,74\}$,
abedc : R^{75},

$\hat{r}$: abcde : $\hat{R}^1$,
and for the rest $\hat{r}$ is the same as r.

By the Borda rule e is chosen at r and a is chosen at $\hat{r}$. Note that a is 1's best alternative. It is clear that, whenever he is confronted with the orderings R^2 up to R^{75} he is better off by claiming that his preference is $\hat{R}^1$ rather than R^1. Hence, this rule is vulnerable for manipulation.

■

The above example leads to the following condition for choice correspondences.

Definition 1.2.4

Let $\Gamma = \langle A,N \rangle$ be a society with $|N| = n$, let $V(A)$ be a set of orderings on A and let C be a choice corresponcence from $V_n(A)$ to $2^A - \{\varphi\}$.

Then C is non-manipulable (or strategy proof), iff for all $i \in N$, all $r, \hat{r} \in V_n(A)$, such that for all $j \in N - \{i\}$ $R^j = \hat{R}^j$, all $x \in C(r)$ and all $y \in C(\hat{r})$: $x \geq y : R^i$.

■

C is non-manipulable, iff a deviation from a given profile r by one individual i to profile $\hat{r}$ is not profitable for i.

It is clear that non-manipulability excludes situations as described in example 1.2.3. Furthermore, it is pointed out here that this non-manipulability condition opens a way to introduce several game theoretical aspects in Social Choice Theory. Gibbard [1973] introduced the game form concept already at the very beginning of the study of non-manipulable choice rules. Maskin [1979] shows the relationship between non-manipulability and Nash equilibria (See also e.g. Moulin [1983] and Peleg [1984]).

Notwithstanding these defects, rules belonging to the class of the Borda rule are frequently used. To this class belong those rules in which alternatives are given a point $s_1, s_2, \ldots$ or s_p, with $s_1 \geq s_2 \geq \ldots \geq s_p \geq 0$, by each individual according to his (her) preference and the alternatives, whose points add up to the greatest sum, are chosen. Those rule are known as score rules. Young [1975] gives a nice characterization of these rules.

If we take $s_1 = 1$, $s_2 = 0, \ldots, s_p = 0$, we are again back to relative majority voting. If $s_1 = p$, $s_2 = p-1, \ldots, s_p = 1$, then the score rule coincides with the Borda rule. For instance, the jury of the Eurovision Song Contest uses a score rule, where $s_1 = 12$, $s_2 = 10$ and so on, to determine the winning song.

We will now answer question 1.

§ 1.3 Pairwise Comparison

One way of simplifying the calculation, for an outcome of a welfare function, is to suppose that this outcome can be obtained from pairwise comparisons of all pairs in A. This can be demonstrated as follows.

Let $\Gamma = \langle A,N \rangle$ be a society, such that $|A| = p$ and $|N| = n$. There are $\binom{p}{2} = \frac{1}{2}p(p-1)$ (not ordered) pairs in A. To demonstrate the complexity of the calculations of outcomes of a welfare function let us agree that 1 comparison between two alternatives in 1 individual ordering stands for 1 unit in the complexity index.

The Borda rule, (in finding the Borda preference), uses for every alternative in every individual preference, all the comparisons with all other alternatives. Hence, its complexity-index is equal to $n \cdot p \cdot \binom{p}{2}$.

A calculation based on pairwise comparisons uses for every individual all the comparisons of one alternative with all other alternatives. Hence, its complexity is equal to $n \cdot \binom{p}{2}$.

Compared to the Borda rule, such a pairwise comparison principle uses less complexity units. But the application of such a rule requires that the following condition holds for the welfare function.

<u>Definition 1.3.1</u> <u>Independence of irrelevant alternatives</u>

Let $\Gamma = \langle A,N \rangle$ be a society, let $V(A)$ and $U(A)$ be sets of orderings on A and let F be a welfare function on Γ from $V_n(A)$ to $U(A)$.

F is <u>independent of irrelevant alternatives</u>, iff for all $x,y \in A$ and all $r, \hat{r} \in V_n(A)$:

$$\text{if } r|_{\{x,y\}} = \hat{r}|_{\{x,y\}}, \text{ then } F(r)|_{\{x,y\}} = F(\hat{r})|_{\{x,y\}}.$$

■

Here $r|_{\{x,y\}} := \langle R^1|_{\{x,y\}}, R^2|_{\{x,y\}}, \ldots, R^n|_{\{x,y\}} \rangle$ and for a relation R on A: $R|_{\{x,y\}} := R \cap (\{x,y\} \times \{x,y\})$. $R|_{\{x,y\}}$ is equal to the relation R restricted to $\{x,y\}$. It is the information of R about $\{x,y\}$. F is independent of irrelevant alternatives, iff for all $x,y \in A$ and all pairs of combinations of individual orderings r and $\hat{r}$, which have the same (ordering) information about x and y ($r|_{\{x,y\}} = \hat{r}|_{\{x,y\}}$), it holds that the corresponding outcomes $F(r)$ and $F(\hat{r})$ have the same (ordering) information about x and y ($F(r)|_{\{x,y\}} = F(\hat{r})|_{\{x,y\}}$). In case that $r|_{\{x,y\}} = \hat{r}|_{\{x,y\}}$ and $F(r)|_{\{x,y\}} \neq F(\hat{r})|_{\{x,y\}}$, it is clear that F uses more information than the information about x and y alone, when determining the preference between them. This explains the name of the condition as well as the fact that pairwise comparison presupposes this condition.

The Borda rule is <u>not</u> independent of irrelevant alternatives. Take r and $\hat{r}$ of example 1.2.2, then

$r|_{\{a,c\}} = \hat{r}|_{\{a,c\}}$, but $\mathrm{Borda}(r)|_{\{a,c\}} \neq \mathrm{Borda}(\hat{r})|_{\{a,c\}}$.

Arrow [1978] introduced this condition. See also e.g. Blau [1971], who introduced some at first sight seemingly weaker independency conditions, which all turn out to be equivalent to the one introduced here (under certain domain conditions that are usually imposed).

Arrow deduced the need for this condition from the pairwise majority rule. Condorcet [1785] encountered already some problems with this rule. The profiles that cause the problem have still his name.

<u>Example 1.3.2</u> <u>Condorcet profiles</u>

Let $\Gamma = \langle\{a,b,c\},\{1,2,3\}\rangle$ be a society.

Take the following profile $r \in L_3(\{a,b,c\})$.

r: a b c : R^1,

b c a : R^2 and

c a b : R^3.

Then a compared to b wins by two votes to one,

b compared to c wins by two votes to one, and

c compared to a wins by two votes to one.

Hence, the pairwise majority rule assigns to r the following relation:
$$\begin{tikzcd} a \arrow[r] & b \arrow[r] & c \arrow[ll, bend left] \end{tikzcd}$$

This is no longer a transitive ordering with a best element. Hence, we do not know what to do now!
The combination r is also known as a Condorcet profile.

■

It is clear that if this pairwise majority rule results in a relation with a best alternative x, this candidate x is a rather good choice. Each counter-objection for another candidate y would be rejected, because x, being the best, beats y in pairwise comparison. This is true in general for every pairwise comparison rule. For that reason Arrow formulated the condition of independence of irrelevant alternatives.

Apart from pairwise majority rules, also other pairwise comparison rules have been introduced. They differ from the pairwise majority rule only when this yields a cyclic preference as in a Condorcet profile. Several of the rules such as the Kramer rule, Copeland rule and Black rule are discussed in e.g. Moulin [1983]. But all these rules have manipulation defects when there are at least three alternatives.

Since every welfare function is automatically independent of irrelevant alternatives in the case of a two alternative set, one may expect that in those societies there is seldom any problem in finding a feasible decision procedure. Usually this is a majority (or voting) rule. See also May [1952] who characterized this type of rule.

Finally it should be noted that Round-Robbin tournaments are essentially welfare functions based on pairwise comparisons.

§ 1.4 Voting by veto

In this section we discuss rules whose choices are determined by sequential elimination of alternatives. We start with the Coombs rule. Its performance is shown by an example; no formal description is given here (See e.g. Moulin [1983]).

Example 1.4.1 Coombs rule

Let $\Gamma = \langle\{a,b,c,d,e\},\{1,2,...,16\}\rangle$ be a society and let $r \in L_{16}(\{a,b,c,d,e\})$ as follows:

r : dcbae : R^1,
dcabe : R^2,
abcde : R^i for $i \in \{3,4\}$,
edcba : R^i for $i \in \{5,6,7\}$,
edcab : R^i for $i \in \{8,9,10\}$,
edabc : R^i for $i \in \{11,12\}$,
edbac : R^{13},
ebacd : R^{14},
ebcad : R^{15} and
eacbd : R^{16}.

According to the Coombs rule, we consecutively eliminate the worst alternative.

Note that e is 4 times the worst alternative in r, and a, b, c and d all 3 times. Therefore, e is first eliminated at r. Next, the worst element of $r|_{X_1}$ is eliminated with $X_1 = A - \{e\}$. In $r|_{X_1}$, a and b are 4 times the worst alternative, c 3 times and d 5 times. Hence d is eliminated. Next, the worst element of $r|_{X_2}$ is eliminated where $X_2 = X_1 - \{d\}$. In $r|_{X_2}$, a and b are 5 times the worst alternative and c 6 times. Hence, c is eliminated. Finally, the worst element of $r|_{X_3}$ is eliminated with $X_3 = X_2 - \{c\}$. This yields to the elimination of b. Therefore, Coombs choice at r is a, and the Coombs social preference at r, Coombs(r), is abcde : Coombs(r).

The Coombs rule eliminates consecutively the worst candidates. Elimination procedures to obtain a good candidate are often used.
Let us indicate one of the most striking defects of such a rule. Consider $r \in L_{16}(\{a,b,c,d,e\})$, where $\hat{R}^i = R^i$ for all $i \in \{2,3,\ldots,16\}$ and dcbea $: \hat{R}^1$. Hence, $\hat{r}$ can be obtained from r by just changing the ordering of one pair of alternatives in the ordering of agent 1.
It is straightforward to calculate that at $\hat{r}$ the Coombs rule consecutively eliminates first a, then b, then c, then d, making e the chosen outcome of the Coombs choice correspondence and edcba : Coombs($\hat{r}$).
But Coombs(r) and Coombs($\hat{r}$) have a completely reversed order and differ in $\binom{p}{2}$ pairs, where r and $\hat{r}$ differ only in 1 pair. This rule is certainly not robust against small changes. On the contrary, it is very sensitive and should therefore only be applied when the individuals are very sure about their individual orderings.
Given the same pair of profiles r and $\hat{r}$, it follows that the Coombs rule is not independent of irrelevant alternatives:

$$r|_{\{a,d\}} = \hat{r}|_{\{a,d\}}, \text{ but Coombs}(r)|_{\{a,d\}} \neq \text{Coombs}(\hat{r})|_{\{a,d\}}.$$

Furthermore, it is straightforward to show that the Coombs choice correspondence is manipulable.

■

Another voting rule is the veto rule. Its performance is demonstrated in the following example.

Example 1.4.2 Application of the Sincere Veto rule

We present only a very special application of veto rules. For more information about veto rules the reader is refered to e.g., Moulin [1983]. These veto rules have nice interpretations in terms of cooperative games.
Let $\Gamma = \langle\{a,b,c,d,e\},\{1,2,3,4\}\rangle$ be a society. To determine the outcome of the veto rule, discussed here, every individual is allowed, by turns, to eliminate one alternative. Since there is one more alternative than candidates, one alternative is left over.

We will suppose that agent 1 starts the elimination procedure, then agent 2 eliminates his candidate and so on, until agent 4. Furthermore, we suppose that the individuals do not know anything about the orderings of the others. It is therefore likely that each eliminates his worst alternative from the alternatives left over at the moment of his turn. This explains the word "sincere". Let us now explain the rule for a given profile $r \in L_4(\{a,b,c,d,e\})$.

Let r be defined as follows:

$abcde : R^i$ for $i \in \{1,2\}$, and

$edcba : R^i$ for $i \in \{3,4\}$.

It is clear that 1 eliminates e, 2 eliminates d, 3 eliminates a and 4 eliminates b, so that c becomes the sincere veto winner at r. $V(r) = \{c\}$. Here V is the choice correspondence which assigns the veto winner at a combination $r \in L_4(\{a,b,c,d,e\})$.

An appreciable property of this veto procedure is that it does not select an alternative which is unacceptable for one of the individuals, since each eliminates his worst alternative. As one can see, for r, that alternative is chosen, which is in the "middle" of all preferences.

On the other hand, it has a defect with respect to a monotonicity property. To illustrate define $\hat{r} \in L_4(\{a,b,c,d,e\})$ as follows:

$cadbe : \hat{R}^i$ for $i \in \{1,2\}$, and

$edcba : \hat{R}^i$ for $i \in \{3,4\}$.

To obtain $\hat{r}$ from r one has to increase in relation 1 and 2 the preference for c, and to decrease the preference for b.

Now at $\hat{r}$ 1 eliminates e, 2 eliminates b, 3 eliminates a and 4 finally eliminates c. Hence, $V(\hat{r}) = \{d\}$.

It follows that although the preference for c has been increased, going from r to $\hat{r}$ and c has been chosen at r, c is not chosen at $\hat{r}$. So V is not monotone according to this preference increment.

■

We end this section with the introduction of two monotonicity conditions: one for choice correspondences and one for welfare functions. In mathematical analysis, a monotone function preserves a certain ordering of the originals in its range. Hence, the function leaves a certain ordering of the originals invariant. The above meant condition leaves such an ordering on the originals, in this case profiles, invariant. Since the ordering which is left invariant, is on profiles, it is desirable to comment somewhat further on these conditions. We will not go very deep, however, because the tool by which this is done, duality, needs a closer investigation. This would lead us too far away from our goal here.

Let us first look at preference changes between two alternatives.

<u>Example 1.4.3</u> <u>Preference changes between two alternatives</u>

Let R_1 and R_2 be two reflexive and complete relations on A, and let $x,y \in A$. Now there are three orderings possible for x and y in R_i ($i \in \{1,2\}$) namely: $x > y : R_i$, $y > x : R_i$ and $x \sim y : R_i$. Hence, there are nine cases possible to describe the ordering of x and y in R_1 and R_2, namely:

<u>Case I</u> $x > y : R_1$ and $x > y : R_2$,
<u>Case II</u> $x > y : R_1$ and $x \sim y : R_2$,
<u>Case III</u> $x > y : R_1$ and $x < y : R_2$,
<u>Case IV</u> $x \sim y : R_1$ and $x > y : R_2$,
<u>Case V</u> $x \sim y : R_1$ and $x \sim y : R_2$,
<u>Case VI</u> $x \sim y : R_1$ and $x < y : R_2$,
<u>Case VII</u> $x < y : R_1$ and $x > y : R_2$,
<u>Case VIII</u> $x < y : R_1$ and $x \sim y : R_2$, and
<u>Case IX</u> $x < y : R_1$ and $x < y : R_2$.

Let us agree upon the following:
when going from R_1 to R_2 there is a change:
(A) in favour of the preference of x to y
strictly in case IV, VII, and VIII, and
weakly in case I, V, and IX,
(B) in favour of the preference of y to x
strictly in case II, III, and VI, and
weakly in case I, V, and IX.

Otherwise stated: R_1 is strictly prefered to R_2 according to $\{\langle x,y\rangle\}$ in case IV, VII and VIII. R_1 is as least as good as R_2 according to $\{\langle x,y\rangle\}$ in case I, IV, V, VII, VIII and IX.

■

By example 1.4.3 it is straightforward to define an ordering on orderings (or combinations of orderings) according to a specific relation.

<u>Definition 1.4.4</u> <u>Preference on orderings</u>

Let $\Gamma = \langle A,N\rangle$ be a society, with $|N| = n$, let $V(A)$ be a set of orderings on A, let R be a relation on A and let r, $\hat{r} \in V_n(A)$.

Then r <u>is as least as much prefered to</u> $\hat{r}$ <u>according to</u> R, iff for all $i \in N$ and for all $\langle x,y\rangle \in R$:

if $\langle x,y\rangle \in R^i$, then $\langle x,y\rangle \in \hat{R}^i$, and

if $\langle y,x\rangle \in \hat{R}^i$, then $\langle y,x\rangle \in R^i$.

■

Notation: $\langle r,\hat{r}\rangle \in \Omega_n(R)$ or $r \geq \hat{r} : \Omega_n(R)$

Instead of $\Omega_1(R)$ also $\Omega(R)$ is written.

After some trivial identifications (i.e., $\langle R^1\rangle$ is identified with R^1) the following becomes apparent:

$R_1 \geq R_2 : \Omega(\{\langle x,y\rangle\})$, i.e., $\langle R_1,R_2\rangle \in \Omega(\{\langle x,y\rangle\})$ in the cases I, IV, V, VII, VIII and IX,

$R_1 > R_2 : \Omega(\{\langle x,y\rangle\})$, i.e., $\langle R_1,R_2\rangle \in \bar{a}\Omega(\{\langle x,y\rangle\})$ in the cases IV, VII and VIII, where $\bar{a}\Omega(\{\langle x,y\rangle\})$ is the asymetric part of $\Omega(\{x,y\})$.

$R_2 \geq R_1 : \Omega(\{\langle x,y\rangle\})$ in the cases I, II, II, V, VI, and IX,

$R_2 > R_1 : \Omega(\{\langle x,y\rangle\})$ in the cases II, III, and VI, and

$R_2 \sim R_1 : \Omega(\{\langle x,y\rangle\})$, i.e., $\langle R_2,R_1\rangle \in \bar{s}\Omega(\{\langle x,y\rangle\})$, in the cases I, V, and IX.

Hence, for a singleton relation $\{\langle x,y\rangle\}$, $\Omega(\{\langle x,y\rangle\})$ coincides with the intuitive agreement on ordering two relations as expressed in example 1.4.3. Furthermore, it is evident that

$\Omega_n(R) = \cap \{\Omega_n(\{\langle x,y\rangle\}) : \langle x,y\rangle \in R\}$. Hence, $\Omega_n(R)$ is a very natural way of extending the definition of the singleton relation case. But this makes it also clear that Ω is in fact a duality operator (See Evers & van Maaren [1985]). This means that the relation between the pairs of profiles of orderings according to a specific relation R and that specific relation is special, not bijective but more special than an average correspondence. Some preliminary results on this subject result in the conclusion that many so called momotonic conditions for welfare functions and choice correpondences, can be formulated by virtue of this duality. Let us give two examples: one concerning positive associativity for welfare functions (See e.g. Ritz [1985] and McManus [1983]) and one concerning strong positive associativity for choice correspondences (See e.g. Moulin [1983] and Satterthwaite & Muller [1977]). These two are the only two monotonicity properties discussed in this monograph. The reader may find a lot of variations to these with respect to welfare functions in McManus [1983] .

<u>Definition 1.4.5</u> <u>Positive associativity</u>

Let $\Gamma = \langle A,N\rangle$ be a society with $|N| = n$, let $V(A)$ and $U(A)$ be sets of orderings on A and let F be a welfare function from $V_n(A)$ to $U(A)$ on Γ.

F is <u>positively associated</u>, iff for all $R \subseteq A \times A$ and all $r,\hat{r} \in V_n(A)$:

$$\text{if } r \geq \hat{r} : \Omega_n(R), \text{ then } F(r) \geq F(\hat{r}) : \Omega(R).$$

■

A welfare function F is positively associated iff F leaves all orderings $\Omega_n(R)$ "invariant", that is, the images are ordered according to $\Omega(R)$. This condition is called here positive association because it is equivalent to the property with the same name often used in literature. To save space and time the proof of this is left to the reader (It is not evident but straightforward using the notions of example 1.4.3). To illustrate the usefullness of this Ω-operator we state the following result:

Theorem 1.4.6

Let $\Gamma = \langle A,N \rangle$ be a society, with $|N| = n$, let $V(A)$ and $U(A)$ be sets of orderings on A and let F be a welfare function on Γ from $V_n(A)$ to $U(A)$. Then (1.4.6.1), (1.4.6.2) and (1.4.6.3) are equivalent.

1.4.6.1 F is independent of irrelevant alternatives.

1.4.6.2 For all $R \subseteq A \times A$, such that $\bar{s}R = R$, and all $r, \hat{r} \in V_n(A)$:

$$\text{if } r \geq \hat{r} : \Omega_n(\bar{s}R), \text{ then } F(r) \geq F(\hat{r}) : \Omega(\bar{s}R).$$

1.4.6.3 For all $R \subseteq A \times A$, and all $r, \hat{r} \in V_n(A)$:

$$\text{if } r \sim \hat{r} : \Omega_n(R), \text{ then } F(r) \sim F(\hat{r}) : \Omega(R).$$

Proof of theorem 1.4.6

The proof of this theorem follows immediately from the following equivalences, which holds for every n and $x,y \in A$:

$$r|_{\{x,y\}} = \hat{r}|_{\{x,y\}}, \text{ iff } \langle r,\hat{r} \rangle \in \Omega_n(\{\langle x,y \rangle,\langle x,y \rangle\}),$$

$$\text{iff } \langle r,\hat{r} \rangle \in \bar{s}\Omega_n(\{\langle x,y \rangle\}).$$

■

By theorem 1.4.6 it is shown that the independence condition is actually also a monotonicity condition. Furthermore, since $\Omega_n(R)$ is transitive, it follows immediately:

Theorem 1.4.7

Let $\Gamma = \langle A,N \rangle$ be a society, let $V(A)$ and $U(A)$ be sets of orderings on A and let F be a positively associated welfare function on Γ from $V_n(A)$ to $U(A)$. Then F is independent of irrelevant alternatives.

■

We will now formulate an equivalent condition for choice correspondences, often called strong positive association.

Definition 1.4.8

Let $\Gamma = \langle A,N \rangle$ be a society, let $V(A)$ be a set of orderings on A and let C be a choice correspondence on Γ from $V_n(A)$ to $2^A - \{\varphi\}$.

C is strongly positively associated, iff for all $r, \hat{r} \in V_n(A)$ and all $x \in C(r)$:

$$\text{if } \hat{r} \geq r : \Omega_n(\{x\} \times A), \text{ then } x \in C(\hat{r}).$$

■

Note that $\langle r,\hat{r}\rangle \in \Omega_n(\{x\} \times A)$, iff in all components $i \in N$ and for all alternatives $y \in A$ the preference of x above y changes weakly in its favour, when going from R^i to $\hat{R}^i$. It is therefore easily seen that x is preserved, going from r to $\hat{r}$. Hence, this condition is equivalent with the one of equal name mentioned in literature (See e.g. Moulin [1983]). To see that this condition is again preserving some ordering `type' it would be necessary to define a relation on 2^A according to a given $R \subseteq A \times A$. This will not be done here, but is possible. (There are some preliminary results about this subject).

Returning to our example 1.4.2, it is clear that the sincere veto rule is not strongly positively associated. Such a defect would not occur if the choice correspondence were strongly positively associated.

§ 1.5 Some other conditions for decision rules

In this section we formulate some other conditions which are frequently imposed on welfare functions or choice correspondences.

The first condition we introduce is called neutrality.

Definition 1.5.1 Neutrality

Let $\Gamma = \langle A,N \rangle$ be a society, with $|N| = n$, let $V(A)$ and $U(A)$ be sets of orderings on A and let S_A be the set of permutations on A.

For $R \subseteq A \times A$ and $\sigma \in S_A$ define:

$$\sigma R := \{\langle x,y \rangle \in A \times A : \langle \sigma^{-1}(x), \sigma^{-1}(y) \rangle \in R\}.$$

For $r \in V_n(A)$ define: $\sigma r := \langle \sigma R^1, \sigma R^2, \ldots, \sigma R^{n1} \rangle$.

A welfare function F on Γ from $V_n(A)$ and $U(A)$ is neutral, iff for all $r \in V_n(A)$ and all $\sigma \in S_A$: $F(\sigma r) = \sigma F(r)$.

A choice correpondence C on Γ from $V_n(A)$ to $2^A - \{\varphi\}$ is neutral, iff for all $r \in V_n(A)$ and all $\sigma \in S_A$:

$$C(\sigma r) = \sigma C(r).$$

■

Since a permutation on A is just a renaming of all alternatives, it follows that neutral decision procedures cannot give a discriminative treatment of a specific alternative in the decision procedure. The procedure applies to all alternatives the same rule. Neutrality is often present in decision procedures, e.g. voting rules, veto rules and the Borda rule are all neutral.

Another way to look at the neutrality property is to interpret it as a simplification condition. The rule becomes more complicated, if it is not neutral, since in that case the rule has to take into account not only the profiles of individual orderings, but also the names of the alternatives, in order to derive a decision.

The second condition introduced here is Pareto-optimality.

Definition 1.5.2 Pareto-optimality

Let $\Gamma = \langle A,N \rangle$ be a society, with $|N| = n$, let $V(A)$ and $U(A)$ be sets of orderings on A, let F be a welfare function on Γ from $V_n(A)$ to $U(A)$ and let C be a choice correpondence on Γ from $V_n(A)$ to $U(A)$.

F is Pareto-optimal, iff for all $r \in V_n(A)$ and $R \in V(A)$:

if for all $i \in$ $R^i = R$, then $\bar{a}R \subseteq \bar{a}F(r)$.

F is strongly Pareto-optimal, iff for all $r \in V_n(A)$ and all $x,y \in A$:

if for all $i \in N$ $x \geq y : R^i$ and there is a $j \in N$, with $x > y : R^j$, then $x > y : F(r)$.

C is Pareto-optimal, iff for all $r \in V_n(A)$ and all $x,y \in A$:

if for all $i \in N$ $x > y : R^i$, then $y \notin C(r)$.

■

Pareto-optimality guarantees a certain degree of autonomy. That is, at several profiles, where there is a unanimous agreement between the individuals a Pareto-optimal decision rule does not defect on that point.

Again, these conditions appear frequently to be fulfilled by decision rules. A rule which is neither Pareto-optimal nor neutral is for instance, a decision rule which assigns to every profile the same outcome (except, in the case of welfare functions, the total indifference relation is excluded to be that outcome).

The following type of condition we introduce is that of non-dictatorship.

Definition 1.5.3 Non-dictatorship

Let $\Gamma = \langle A,N \rangle$ be a society with $|N| = n$, let $V(A)$ and $U(A)$ be sets of orderings on A and let F be a welfare function on Γ from $V_n(A)$ to $U(A)$.

F is strongly non-dictatorial, iff for all $i \in N$ there are $x,y \in A$ and $r \in V_n(A)$, such that:

$x > y : R^i$ and $y > x : F(r)$.

F is weakly non-dictatorial, iff for all $i \in N$ there are $x,y \in A$ and $r \in V_n(A)$, such that:

$x > y : R^i$ and $y \geq x : F(r)$.

■

If a welfare function F is strongly (weakly) non-dictatorial, then, for every individual $i \in N$, there exists a profile r such that, at this profile, the society decides strictly against (not in favour of) i with respect to at least one pair of alternatives. If F is not strongly (weakly) non-dictatorial, then there is a <u>weak</u> (<u>strong</u>) <u>dictator</u> i such that for every $r \in V_n(A)$: $\bar{a}R_i \subseteq F(r)$ $(\bar{a}R_i \subseteq \bar{a}F(r))$.

Certainly a dictatorial rule is of a simple nature. An important weak dictatorial rule is that of <u>unanimity</u>:

$F : W_n(A) \to Q(A)$ (= the set of quasi-orderings), defined as follows: $\langle x,y\rangle \in F(r)$, iff for all $i \in N$ $\langle x,y\rangle \in R^i$.
This unanimity rule F yields an incomparability between two alternatives whenever the individuals are conflicting (disagree in preference) about these alternatives. In this rule every one is a weak dictator. F is not strongly dictatorial.

Finally we introduce a single-valuedness condition.

<u>Definition 1.5.4</u> <u>Single-valuedness</u>

Let $\Gamma = \langle A,N\rangle$ be a society, with $|N| = n$, and let $V(A)$ be a set of orderings.
A choice correspondence C on Γ from $V_n(A)$ to $2^A - \{\varphi\}$ is <u>single-valued</u>, iff for all $r \in V_n(A)$: $|C(r)| = 1$.

■

Single-valued choice correspondences can be identified with functions from $V_n(A)$ to A. They are often called choice functions.

All the conditions defined here are well known in literature (See e.g. Moulin [1983], Sen [1970] or Sen [1986]). In fact this collection exposed here is only a small fraction of all conditions often used.

§ 1.6 Decisiveness

In the foregoing sections several conditions on welfare functions have been introduced. Looking at the formulation of these conditions it is apparent that they are described by a wide variety of formulas. Since we are interested in the effects that one conditions may have on another, it is necessary to have a common language for all of these conditions. As the original interpretations become less transparant in this `new' language, this language has not been used to introduce the conditions in the foregoing sections.

The language proposed here is inspired by Kelly [1978], although here his total vocabulary is not used. That what is used is transformed into notations which are suitable here and for the rest of this monograph. The language has a decisiveness interpretation.

Definition 1.6.1 Decisiveness

Let $\Gamma = \langle A,N \rangle$ be a society, let $S \subseteq N$, let $V(A)$ and $U(A)$ be sets of orderings on A, let $r \in V_n(A)$, let $x,y \in A$ and let F be a welfare function on Γ from $V_n(A)$ to $U(A)$.

1.6.1.1 The set of unanimously feasible pairs by S in $V_n(A)$ is

$K(V_n(A),S) := \{\langle x,y \rangle \in A \times A$: There is a profile $r \in V_n(A)$, such that for all individuals $i \in S$: $x > y : R^i\}$.

1.6.1.2 S is (quasi-)decisive at r and F on $\langle x,y \rangle$, iff
if $\langle x,y \rangle \in K(V_n(A),S)$ and
for all $i \in S : x > y : R^i$ (and
for all $i \in N\text{-}S : y > x : R^i$),
then $x > y: F(r)$.
Notation: $(q)D(F,S,\langle x,y \rangle,r)$.

1.6.1.3 S is (quasi-)blocking at r and F on $\langle x,y \rangle$, iff
if $\langle x,y \rangle \in K(V_n(A),S)$ and
for all $i \in S : x > y : R^i$ (and
for all $i \in N\text{-}S : y > x : R^i$),
then not $y > x : F(r)$.
Notation: $(q)B(F,S,\langle x,y \rangle,r)$.

1.6.1.4 S is (quasi-)decisive at F on $\langle x,y\rangle$, iff for all $r \in V_n(A)$: $(q)D(F,S,\langle x,y\rangle,r)$. Notation: $(q)D(F,S,\langle x,y\rangle)$.

1.6.1.5 S is (quasi-)blocking at F on $\langle x,y\rangle$, iff for all $r \in V_n(A)$: $(q)B(F,S,\langle x,y\rangle,r)$. Notation: $(q)B(F,S,\langle x,y\rangle)$.

1.6.1.6 S is (quasi-)blocking at F, iff for all $\langle a,b\rangle \in K(V_n(A),S)$: $(q)B(F,S,\langle a,b\rangle)$.

1.6.1.7 S is (quasi-)decisive at F, iff for all $\langle a,b\rangle \in K(V_n(A),S)$: $(q)D(F,S,\langle a,b\rangle)$.

■

When S is decisive at r and F on $\langle x,y\rangle$, it follows that, if x is unanimously strictly prefered to y by S at r, then society prefers x to y at r, i.e., $x > y : F(r)$. Hence, at r it seems that S decides about x and y. S is quasi-decisive at r and F on $\langle x,y\rangle$, if in S everyone prefers x to y (strictly) and outside S everyone prefers y to x, then $x > y : F(r)$. Hence, quasi-decisiveness is just decisiveness restricted to special profiles of the orderings of individuals in N - S. All the other notions are just derivates of these two.

Let $(q)D(F,S) := \{\langle x,y\rangle \in K(V_n(A),S) : (q)D(F,S,\langle x,y\rangle)\}$ and $(q)B(F,S) := \{\langle x,y\rangle \in K(V_n(A),S) : (q)B(F,S,\langle x,y\rangle)\}$.

We have now the following theorem which characterizes several conditions in terms of decisiveness. This enables us to produce deductions from sets of conditions jointly imposed on welfare functions, which is done in chapter 4.

Theorem 1.6.2

Let $\Gamma = \langle A,N\rangle$ be a society, with $|N| = n$, let V(A) be a set of complete, reflexive and antisymmetric relations on A, let U(A) be a set of reflexive and complete relations on A and let F be a welfare function from $V_n(A)$ to U(A) on Γ.

1.6.2.1 F is independent of irrelevant alternatives, iff for all $S \subseteq N$ and $\langle a,b\rangle \in K(V_n(A),S) \cap \bar{v}K(V_n(A),N-S)$: if $\langle b,a\rangle \notin qB(F,N-S)$, then $\langle a,b\rangle \notin qD(F,S)$.

Furthermore, let F be independent of irrelevant alternatives.

1.6.2.2 F is Pareto-optimal, iff $D(F,N) = qD(F,N) = K(V_n(A),N)$.

1.6.2.3 F is strongly non-dictatorial, iff for all $i \in N$:

$qD(F,N-\{i\}) \neq \varphi$.

1.6.2.4 F is weakly non-dictatorial, iff for all $i \in N$:

$qB(F,N-\{i\}) \neq \varphi$.

1.6.2.5 F is neutral, iff for all $S \subseteq N$:

if $qD(F,S) \cap K(V_n(A),S) \cap \bar{v}(K(V_n(A),N-S) \neq \varphi$,

then $K(V_n(A),S) \cap \bar{v}(K(V_n(A),N-S) \subseteq qD(F,S)$,

(where $\bar{v}R = \{\langle y,x\rangle : \langle x,y\rangle \in R\}$).

<u>Proof of theorem 1.6.2</u>

(1.6.2.1) (only if) Suppose F is independent of irrelevant alternatives, $S \subseteq N$, $\langle a,b\rangle \in K(V_n(A),S) \cap \bar{v}K(V_n(A),N-S)$ and $\langle b,a\rangle \notin qB(F,N-S)$.

Then there is a $r \in V_n(A)$, such that $a > b : R^i$ for $i \in S$, $b > a : R^i$ for $i \in N-S$ and $a > b : F(r)$.

By the independence condition we have for all $\hat{r} \in V_n(A)$:

if for all $i \in S$ $a > b : \hat{R}^i$, and all $i \in N-S$ $b > a : \hat{R}^i$, then $a > b : F(\hat{r})$.

Hence, $\langle a,b\rangle \in qD(F,S)$.

(if) Is of the same simple nature and therefore left to the reader.

(1.6.2.2), (1.6.2.3) and (1.6.2.4) are simple to prove.

(1.6.2.5) (only if) Suppose F is neutral and

$\langle x,y\rangle \in qD(F,S) \cap K(V_n(A),S) \cap \bar{v}K(V_n(A),N-S)$.

Then there is a profile $r \in V_n(A)$, such that:

$x > y : R^i \quad i \in S$,

$y > x : R^i \quad i \in N-S$ and

$x > y : F(r)$.

Take $\langle a,b\rangle \in K(V_n(A),S) \cap \bar{v}K(V_n(A),N-S)$.

Obviously it follows that there is a $\hat{r} \in V_n(A)$, such that:

$a > b : \hat{R}^i \quad i \in S$ and

$b > a : \hat{R}^i \quad i \in N-S$.

Take $\sigma \in S_A$, such that $\sigma(a) = x$, $\sigma(x) = a$, $\sigma(y) = b$, $\sigma(b) = y$ and $\sigma(z) = z$ for all $z \in A - \{x,y,a,b\}$.

Since $F(\hat{r})|_{\{a,b\}} = F(\hat{r}|_{\{a,b\}}) = F(\sigma r|_{\{x,y\}}) = \sigma(F(r)|_{\{x,y\}})$ and $x > y : F(r)$, it follows that $a > b : F(\hat{r})$.

(if) Let $x > y : R^i$ and $a > b : \hat{R}^i$ for $i \varepsilon S$, and

$y > x : R^i$ and $b > a : \hat{R}^i$ for $i \varepsilon N\text{-}S$.

It is sufficient to prove:

$x \geq y : F(r)$, iff $a \geq B : F(\hat{r})$.

This is obviously done by our assumption.

■

The theorem used above will extensively be used in our impossibility theorems of chapter 4. As indicated before it allows us to compare several conditions and deduce some inferences between them. One of these inferences becomes quite obvious namely the impossibility of Sen's minimal liberalism. This criterion imposes a priori several decisiveness properties on coalitions. This conflicts of course with the transitivity of a social preference of a society, because decisiveness is not defined in relation with this transitivity (See e.g., Sen [1970] and Breyer [1978]).

§ 1.7 Organization of this Monograph

Since the path breaking work of Arrow in 1951, it becomes common knowledge that there does not exist a constitution or decision rule that is both generally applicable and not contaminated by serious defects. This knowledge is based on the so called impossibility theorems, many of which have been derived at this moment. Considering conditions that may be imposed on choice correspondences or welfare functions or their domains and ranges, social choice theory studies the effects such conditions may have for the existence of such choice correspondences and welfare functions. Although this theory is formal it is astonishing that within this theory no theoretical attention is paid to one of its main materials: the notion of ordering.

In the following chapter a theoretical background for this notion is developed. It is called a classification system for orderings. This classification system provides a format of primitive concepts in which the usual domains and ranges of decision rules can be fit. Since it is possible to formulate and to prove most of the existing impossibility theorems in this classification system, we are able to come close to the common roots of these theorems and to arrive at the very fundamentals of the theory of social choice. We will formalize the intuition behind the impossibility theorems in chapter 4.

In chapter 3 topologies on discrete metric spaces are introduced. These topologies enable a non-trivial continuity concept for functions between such discrete metric spaces. By virtue of the continuity concept a meaningful weakening of the independence of irrelevant alternatives can be discussed in chapter 4.

Chapters 2 and 3 are both preparing for chapter 4. In this final chapter it is first shown that the most frequently studied type of welfare function is in fact a morphism, which leaves the ordering properties introduced in chapter 2 invariant. This result may motivate the reader to study our classification system.

Next it is shown that there exists a strong relation between welfare functions and choice correspondences, both belonging to the category of decision rules. Again, in this correspondence the above described morphisms play an important rôle.

In section 4.3 we investigate possibilities to weaken conditions on the range of a mapping. Here the usefulness of chapter 2 is demonstrated as indicated above.

In section 4.4 the independence of irrelevant alternatives condition is replaced by a weaker continuity condition. There exist welfare functions with nice properties that are continuous, but not independent of irrelevant alternatives. However, the same type of impossibility theorems occur if some reasonable range restrictions are imposed.

In the final section 4.5 weakenings of the domain conditions are studied. These conditions are of the following type: the set of profiles is not required to be the set of all profiles over the set of linear orderings or weak orderings. That is, the set of profiles might be a real subset of those sets. In literature it is often called a restricted domain condition. In this case it appears that several types of restrictions lead to the existence of nice welfare functions. The descriptions of these restrictions, however, do not result into a transparant set of admissible profiles.

CHAPTER 2 A CLASSIFICATION SYSTEM FOR ORDERINGS

§ 2.1 Introduction

In literature many 'types' of preference orderings have been introduced, e.g., linear orderings, weak orderings, semi-orderings, partial orderings, interval orderings, quasi-orderings, acyclical orderings, tournaments and many less well-known orderings. These orderings have been developed in various fields such as economy, psychology, sociology, operations research, decision theory, discrete mathematics and many others. A lot of research on these orderings was dedicated to model these types of relations and to compare them with each other as well as with graphtheoretical or combinatorial concepts. However, to the best of our knowledge, there has not been an investigation into a system which models all well-known types of orderings. Of course, all orderings have been introduced by imposing criteria on relations in such a way that each type of ordering is determined by its own special set of criteria. Although such a set of criteria can formalize the phenomenon of "linear", "weak", "semi", "partial" "interval" or any other type of ordering one might think of, it does not formalize the phenomenon of "ordering" itself. Hence, we have no formal criteria which enable us to classify a set of relations as a set of orderings. Such criteria may exist since we use the word "ordering" in a specific way whenever we deal with relations.

In chapter 4 we will investigate formalizations of constitutional decision procedures in which various types of orderings play a vital rôle. We will introduce a classification system which formalizes the notion of "ordering". Of course, such a classification system can be used in several other formal theories mentioned above.

We use the word "classification" system rather than "axiom"-system, since an axiom system should be precise in all its definitions, theorems and proofs of these theorems. We will not be exhaustively precise in the definitions, theorems and their proofs, but try to find a good balance between precision and clearness. It is straightforward, although cumbersome, to make precise the system introduced in the next sections such that one could call it an axiomatisation of orderings.

The classification system classifies sets of relations as sets of orderings. Hence, the system consists of criteria for sets of relations. According to those criteria a set of relations can be classified as a set of orderings or not. However, the criteria imposed on linear, weak orderings, etc. are criteria for relations and not for sets of relations. We will end this section by explaining this approach. The decision that a certain relation R is a linear, weak or semi-ordering etc... is based on the fact that R belongs to a class of relations called respectively the set of linear, weak and semi-orderings etc... Hence, the decision whether R is an ordering or not depends on the fact whether that relation R belongs to any class, which can be seen as a set of special orderings. Thus we have to determine how a set of relations can be classified as a set of orderings. This is why we introduce criteria for sets of relations instead of criteria for relations.

This chapter is organized as follows: In section 2 we introduce the classification system and some general results. In section 3 we study linear orderings and the phenomenon of transitivity. In section 4 the minimal extensions of the linear orderings are studied and in the following two sections extensions of these extensions are investigated. Finally, in section 7 we will dwell upon the criteria to discover how slight changes of these criteria may disturb the system.

§ 2.2 The Classification Model.

In this section we introduce the model by which the preference orderings are classified in the following sections. Furthermore, we will recall some well-known operations and state some preliminary results.

Let us start with the model in which we will study relations. First we state explicitly a restriction on the field of temporary interest: Throughout this chapter let U be a countable and infinite set. U is the 'universe' of alternatives on which preference relations will be defined. Since only finite relations are studied here, the possible domains which are finite are of interest to us.

Let $Æ := \{ X \in 2^U: X$ is nonempty and finite$\}$. Æ is the set of finite and non-empty subsets of U. Æ is the set of all possible domains of relations in which we are interested. The set of all possible relations in which we are interested is:

$Å := \{\langle R,X\rangle: X \in Æ \ \& \ R \subseteq X\mathrm{x}X\}$.

An element $\langle R,X\rangle$ of Å is called a <u>relation</u> R (on X). In literature the domain of a relation is usually known and does not vary. In this approach here, however, several operations on a relation by which the domain might change are introduced. For this reason the relation and its domain are explicitly mentioned.

Notation: Let R be a relation on X ($\langle R,X\rangle \in Å$).

Instead of $\langle R,X\rangle$ we also write R_X. In (1.1.4) several notations, definitions and interpretations concerning orderings have been introduced. If we replace R by R_A there, then they become meaningful here. To save space we pretend that this substitution took place and use these notations in this chapter.

To simplify the picture of a graph no edges are drawn from a dot to itself. The context shows whether or not for an element x it holds that: $\langle x,x\rangle \in R_A$.

Now some operations on relations are recalled. Let S_U be the set of permutations on U. Hence, S_U is equal to $\{\sigma : \sigma$ is a bijective function from U to U$\}$. Permutations will be indicated most of the time by the small Greek letters σ and τ. The identity permutation is indicated by $\bar{i}$. Hence, $\bar{i} \in S_U$ such that for all $x \in U$: $\bar{i}(x)=x$.

Furthermore, we remark that relations will be denoted by capital "R" possibly together with some sub- or superscripts and that their domains are denoted by capitals. Operations on relations will be denoted by small letters superscribed by a bar or by special mathematical symbols different from the usual alfabet.

<u>Definition 2.2.1</u> <u>Binary operations on relations</u>

Suppose: $A, B \in$ Æ, $R_A, R_A^1, R_A^2, R_B^3 \in$ Å and $A \cap B = \varphi$.

2.2.1.1 $R_A^1 \cup R_A^2 := \langle\{\langle x,y\rangle \in A \times A: \langle x,y\rangle \in R_A^1$ or $\langle x,y\rangle \in R_A^2\}, A\rangle$ is the <u>union</u> of R_A^1 <u>and</u> R_A^2.

2.2.1.2 $R_A^1 \cap R_A^2 := \langle\{\langle x,y\rangle \in A \times A: \langle x,y\rangle \in R_A^1$ and $\langle x,y\rangle \in R_A^2\}, A\rangle$ is the <u>intersection</u> of R_A^1 <u>and</u> R_A^2.

2.2.1.3 $R_A^1 \circ R_A^2 = \langle\{\langle x,y\rangle \in A \times A:$ there is an element z in A, such that $\langle x,z\rangle \in R_A^2$ and $\langle z,y\rangle \in R_A^1\}, A\rangle$ is the <u>composition</u> of R_A^1 <u>with</u> R_A^2.

2.2.1.4 $[R_A]^1 := R_A$.

$[R_A]^{k+1} := [R_A]^k \circ R_A$ for all $k \geq 1$.

Hence, $[R_A]^k = \underbrace{R_A \circ R_A \circ \dots \circ R_A}_{k \text{ times}}$.

2.2.1.5 $R_A^1 \gg R_B^3 := \langle\{\langle x,y\rangle \in (A \cup B) \times (A \cup B):$ $\langle x,y\rangle \in R_A^1$ or $\langle x,y\rangle \in R_B^3$ or $\langle x,y\rangle \in A \times B\}, A \cup B\rangle$ is called the <u>concatenation</u> of R_A^1 <u>with</u> R_B^3.

■

Since evidently $R_A^1 \cup R_A^2$, $R_A^1 \cap R_A^2$, $R_A^1 \circ R_A^2$ and $R_A^1 \gg R_A^2$ are in Å, $\cup$, $\cap$, $\circ$, and $\gg$ are binary operations on Å. Rosenstein [1982] calls this concatenation operation 'summation'and Jónsson [1982] calls it 'lexicographic product'. Let us illustrate the name `concatenation' for this operation.

Example 2.2.2

Suppose: $A = \{x,y,z\}$, $B = \{a,b\}$, $A \cap B = \varphi$, $A,B \in Æ$ and R_A^1, R_A^2, R_B^3, $R_B^4 \in Å$, with $xyz : R_A^1$

(So $R_A^1 = \langle\{\langle x,y\rangle,\langle x,z\rangle,\langle y,z\rangle\},A\rangle$), $(xy)z : R_A^2$, $ab : R_B^3$ and $\binom{a}{b} : R_B^4$.

Then $xyzab : R_A^1 \gg R_B^3$, $abxyz : R_B^3 \gg R_A^1$

$xyz\binom{a}{b} : R_A^1 \gg R_B^4$ and $(xy)z\binom{a}{b} : R_A^2 \gg R_B^4$.

■

We observe that the concatenation operation is not commutative and furthermore that the representation of the concatenated relation is equal to the concatenation of the representations of each of its argument relations.

Note that the introduced binary operations are associative and that union as well as intersection is commutative, where composition and concatenation is not. Furthermore, Å is closed with respect to all of these operations. Two special relations on a set $A \in Æ$ are the <u>identity relation</u> on A, indicated by Id_A and the empty relation indicated by φ_A.

$Id_A := \langle\{\langle x,x\rangle : x \in A\},A\rangle$ and $\varphi_A := \langle\varphi,A\rangle$.

Now we introduce some monadic operations.

Definition 2.2.3 Monadic operations on relations

Suppose: $A \in Æ$, $R_A \in Å$, $B \subseteq A$, with $B \neq \varphi$, and $\sigma \in S_U$.

2.2.3.1 $\bar{v}R_A := \langle\{\langle x,y\rangle:\langle y,x\rangle \in R_A\},A\rangle$ is the <u>converse</u> of R_A.

2.2.3.2 $\bar{c}R_A := \langle\{\langle x,y\rangle:\langle x,y\rangle \notin R_A \;\&\; x \in A \;\&\; y \in A\},A\rangle$ is the <u>(relative) complement</u> of R_A.

2.2.3.3 $\bar{r}R_A := R_A \cup Id_A$ is the <u>reflexive closure</u> of R_A.

2.2.3.4 $\sigma R_A := \langle\{\langle\sigma(a),\sigma(b)\rangle:\langle a,b\rangle \in R_A\}, \sigma(A)\rangle$ is the σ <u>permutation</u> of R_A.

2.2.3.5 $\bar{s}R_A := R_A \cap \bar{v}R_A$ is the <u>symmetric part</u> of R_A.

2.2.3.6 $\bar{a}R_A := R_A \cap \overline{cv}R_A$ is the <u>asymmetric part</u> of R_A.

2.2.3.7 $R_{A|_B} := \langle\{\langle x,y\rangle \in B \times B : \langle x,y\rangle \in R_A\},B\rangle$ is the <u>restriction</u> of R_A <u>to</u> B.

2.2.3.8 $\bar{t}R_A := \cup\{[R_A]^k : k \in \{1,2,3,\ldots\}\}$ is the <u>transitive closure of</u> R_A.

2.2.3.9 $\bar{d}R_A := R_A \cap Id_A$ is <u>the diagonal part</u> of R_A.
2.2.3.10 $\bar{n}R_A := R_A \cap \bar{c}Id_A$ is <u>non-diagonal part</u> of R_A.
2.2.3.11 $\bar{i}R_A := R_A$ is the <u>identity operation</u> on Å.
2.2.3.12 $\bar{o}R_A := \varphi_A$ is the <u>"constant" empty operation</u> on Å.
2.2.3.13 $\bar{e}R_A := Id_A$ is the <u>"constant" identity operation</u> on Å.
2.2.3.14 $\bar{q}R_A := (R_A \cap \bar{c}Id_A) \cup (\bar{c}R_A \cap Id_A)$ is <u>the relative diagonal complement</u> of R_A.
2.2.3.15 $\bar{m}R_A := (\bar{a}R_A \cup \bar{d}R_A)$ is the <u>antisymmetric part</u> of R_A.
2.2.3.16 Let f_1 and f_2 be two monadic operations on relations, then for all $R_A \in$ Å:

f_1f_2, defined by $f_1f_2R_A := f_1(f_2R_A)$, is the <u>composition</u> of f_1 and f_2,

$f_1 \cup f_2$, defined by $(f_1 \cup f_2)R_A := f_1R_A \cup f_2R_A$, is the <u>union</u> of f_1 and f_2,

$f_1 \cap f_2$, defined by $(f_1 \cap f_2)R_A := f_1R_A \cap f_2R_A$, is the <u>intersection</u> of f_1 and f_2.

■

Notice that Å is closed with respect to all of these operations.

Furthermore, we write $\overline{ne}$ instead of $\bar{n}\bar{e}$, $\overline{qos}$ instead of $\bar{q}\bar{o}\bar{s}$ and so on.

Observing the monadic operations introduced above, we can point out at least three different types of monadic operations with respect to the information which is used in their definition:

A. Monadic operations using information which is not completely contained in the relations which are the arguments of these operations. These operations are defined by virtue of information which is not 'known' to the arguments of these operations, e.g., the permutation operation σ and the restriction operation $|_B$.

B. Monadic operations using information which is completely contained in the argument relations but of a global nature. In such an operation, say f, information, which is not completely contained in the pairs of elements <x,y> with respect to the argument relation R_A, is used to define $\langle x,y\rangle \in fR_A$. Hence, f uses global information of R_A. $\bar{t}$ is such an operation.

C. Monadic operations using only local information. In such an operation, say f, only information concerning the pairs of elements $<x,y>$ in $A \times A$ with respect to an argument relation R_A is used. Hence, only information of $R_{A|_{\{x,y\}}}$ and $\{x,y\}$ is used to define $<x,y> \varepsilon fR_A$.

Such information, only concerning pairs of elements and relations, will be called <u>local information</u> of a pair with respect to a relation, or local information for short . These monadic operations define their images by means of local information about the pairs of elements. $\bar{i}$,$\bar{c}$, $\bar{r}$ and $\bar{v}$ are of this type.

In the next section we will use these monadic operations, which only use local information, in order to describe a general transitivity property; therefore we study this type of operation more carefully here. First we investigate the notion of local information.

Local information about a pair $<x,y> \varepsilon A \times A$ and a relation $R_A \varepsilon \mathring{A}$ is a truthfunctional combination of atomic propositions about $<x,y>$ and R_A. We have precisely the following atomic propositions:

P_1: $x = y$, x is equal to y,

P_2: $<x,y> \varepsilon R_A$, x is in relation R_A with y, and

P_3: $<y,x> \varepsilon R_A$, y is in relation R_A with x.

These atomic propositions yield the following atomic monadic operations:

$m_1R_A := <\{<x,y> \varepsilon AxA : x=y\},A> = \bar{e}R_A$,

$m_2R_A := <\{<x,y> \varepsilon AxA : <x,y> \varepsilon R_A\},A> = \bar{i}R_A$, and

$m_3R_A := <\{<x,y> \varepsilon AxA : <y,x> \varepsilon R_A\},A> = \bar{v}R_A$.

Now it is obvious that every truthfunctional of these atomic propositions corresponds uniquely to a monadic operation which is based on local information. Hence, we are interested in the possible compositions of these atomic propositions. Note that P_1, P_2 and P_3 yield to the following eight combinations of these atomic propositions:

Q_1: P_1 & P_2 & P_3,

Q_2: P_1 & $\neg P_2$ & $\neg P_3$,

Q_3: $\neg P_1$ & P_2 & P_3,

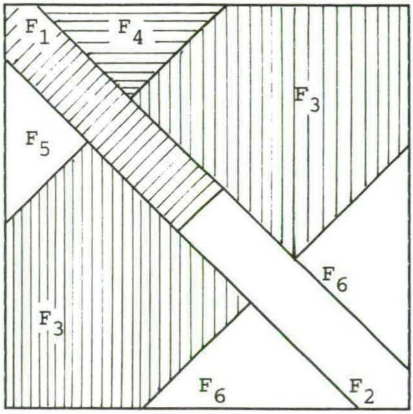
Q_4: $\neg P_1$ & P_2 & $\neg P_3$,
Q_5: $\neg P_1$ & $\neg P_2$ & P_3,
Q_6: $\neg P_1$ & $\neg P_2$ & $\neg P_3$,
Q_7: P_1 & $\neg P_2$ & P_3, and
Q_8: P_1 & P_2 & $\neg P_3$.

Since $\langle y,x\rangle = \langle x,y\rangle$ whenever $x = y$, neither Q_7 nor Q_8 can be fulfilled. The other six can so that there are precisely $2^6 = 64$ truthfunctional operations on P_1, P_2 and P_3. So it is obvious that there are precisely 64 monadic operations which are based on local information. Moreover, from the arguments above it is easy to describe the operations corresponding to Q_1 up to Q_6:

$$f_1R_A = \langle\{\langle x,y\rangle \in AxA : x = y \ \& \ \langle x,y\rangle \in R_A \ \& \ \langle y,x\rangle \in R_A\},A\rangle$$
$$= (\bar{e} \cap \bar{i} \cap \bar{v})R_A = \bar{d}R_A,$$
$$f_2R_A = \langle\{\langle x,y\rangle \in AxA : x = y \ \& \ \langle x,y\rangle \notin R_A \ \& \ \langle y,x\rangle \notin R_A\},A\rangle$$
$$= (\bar{e} \cap \bar{c} \cap \overline{cv})R_A = \overline{dc}R_A,$$
$$f_3R_A = \langle\{\langle x,y\rangle \in AxA : x \neq y \ \& \ \langle x,y\rangle \in R_A \ \& \ \langle y,x\rangle \in R_A\},A\rangle$$
$$= (\overline{ce} \cap \bar{i} \cap \bar{v})R_A = \overline{ns}R_A,$$
$$f_4R_A = \langle\{\langle x,y\rangle \in AxA : x \neq y \ \& \ \langle x,y\rangle \in R_A \ \& \ \langle y,x\rangle \notin R_A\},A\rangle$$
$$= (\overline{ce} \cap \bar{i} \cap \overline{cv})R_A = \bar{a}R_A,$$
$$f_5R_A = \langle\{\langle x,y\rangle \in AxA : x \neq y \ \& \ \langle x,y\rangle \notin R_A \ \& \ \langle y,x\rangle \in R_A\},A\rangle$$
$$= (\overline{ce} \cap \bar{c} \cap \bar{v})R_A = \overline{av}R_A, \text{ and}$$
$$f_6R_A = \langle\{\langle x,y\rangle \in AxA : x \neq y \ \& \ \langle x,y\rangle \notin R_A \ \& \ \langle y,x\rangle \notin R_A\},A\rangle$$
$$= (\overline{ce} \cap \bar{c} \cap \overline{cv})R_A = \overline{nsc}R_A.$$

Notice that if g and h are monadic operations based on local information which correspond to propositions respectively G and H, then $g \cap h$ corresponds to G & H, $\bar{c}g$ corresponds to $\neg G$ and $g \cup h$ corresponds to G v H. Hence, the set of possible monadic operations based on local information is equal to the set of possible unions of the operations f_1 up to f_6, and, of course also equal to the set of all operations, built from m_1, m_2 and m_3 by $\cap$, $\cup$ and $\bar{c}$.

The following pictorial interpretations might be very helpful to understand more easily the theorems and definitions of this chapter.



The diagonal of A x A

The square represents the cartesian product A x A and the shaded area an arbitrary relation R_A on A. R_A induces a partition F_1, F_2, F_3, F_4, F_5 and F_6 of A x A. Some parts consist of two disconnected area's.

$F_1 = f_1R_A = \bar{d}R_A$ is the diagonal part of R_A.

$F_2 = f_2R_A = \overline{dc}R_A$ is the relative diagonal complement part of R_A.

$F_3 = f_3R_A = \overline{ns}R_A$ is the symmetric part of R_A, which is not on the diagonal.

$F_4 = f_4R_A = \bar{a}R_A$ is the asymmetric part of R_A.

$F_5 = f_5R_A = \overline{av}R_A$ is the conversed asymmetric part of R_A.

$F_6 = f_6R_A = \overline{nsc}R_A$ is the incomparable part of R_A.

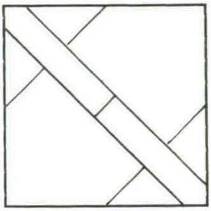
Let $\hat{M}$ be the set of all maonadic operations based on local information. Then $\hat{M} = \{h: h = \bigcup_{1 \leq j \leq 6} g_j$ where $g_1, g_2, .. g_6 \in \{\bar{o}, f_1, .. f_6\}\}$ Observe $\ulcorner \hat{M}, \cup, \cap, \bar{c} \urcorner$ is a Boolean-algebra. In the following example we will examine the elements of $\hat{M}$.

<u>Example 2.2.4</u> <u>Elements of $\hat{M}$</u>

$\hat{M}$ consists of all possible unions over f_1, f_2, f_3, f_4, f_5 and f_6. Let us systematically explore these unions.

2.2.4.0 <u>Empty union</u>: $\cup \{f_i : i \in \varphi\}$.

This union is equal to $\bar{o}$ since the proposition corresponding with this union cannot be fulfilled.

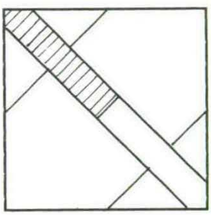
$\bar{0}R_A$: 

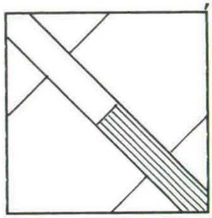
If m is a monadic operator, then mR_A is shaded.

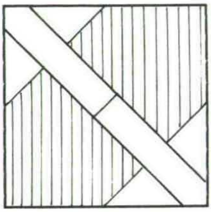
2.2.4.1 <u>Unions of precisely one argument</u>:

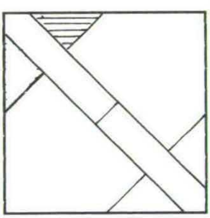
$\cup\{f_i : i \in K\}$, where $|K| = 1$.

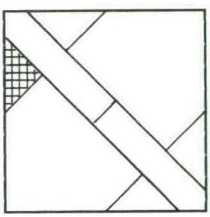
This union is precisely $\{\bar{d}, \overline{dc}, \overline{ns}, \bar{a}, \overline{av}, \overline{nsc}\}$.

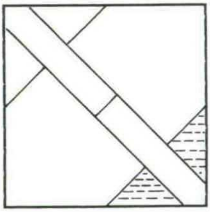
$\bar{d}R_A$: 

, $\overline{dc}R_A$: 

, $\overline{ns}R_A$:  ,

$\bar{a}R_A$: 

, $\overline{av}R_A$: 

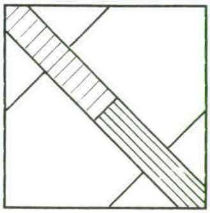
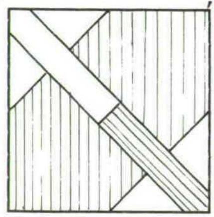
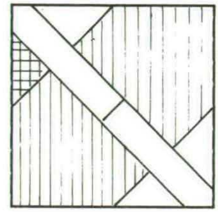
, $\overline{nsc}R_A$:  .

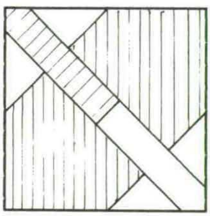
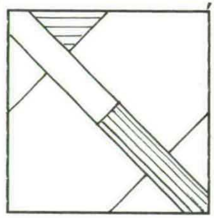
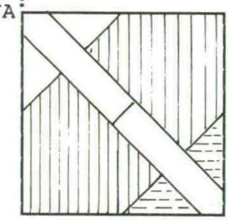
2.2.4.2 <u>Unions of precisely two arguments</u>:

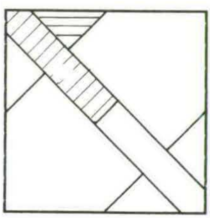
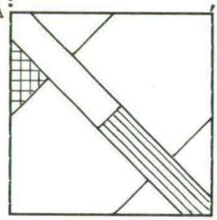
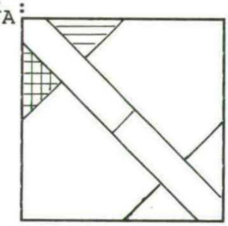
$\cup\{f_i : i \in K\}$, where $|K| = 2$.

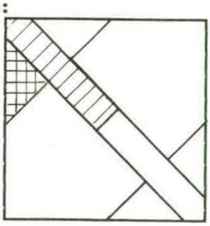
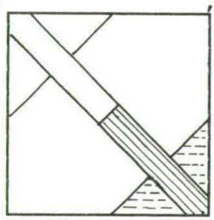
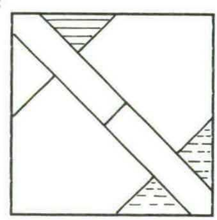
There are $\binom{6}{2} = 15$ of those unions. We will enumerate them:

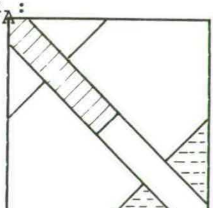
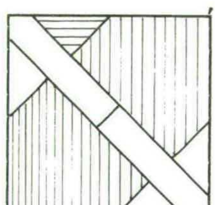
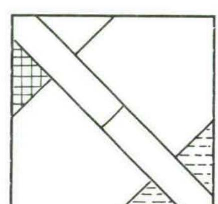
$\bar{d} \cup \overline{dc} = \bar{e}$,	$\overline{dc} \cup \overline{ns} = \overline{qs}$,	$\overline{ns} \cup \overline{av} = \overline{nv}$,
$\bar{d} \cup \overline{ns} = \bar{s}$,	$\overline{dc} \cup \bar{a} = \overline{qm}$,	$\overline{ns} \cup \overline{nsc} = \overline{nsca}$,
$\bar{d} \cup \bar{a} = \bar{m}$,	$\overline{dc} \cup \overline{av} = \overline{qmv}$,	$\bar{a} \cup \overline{av} = \overline{csca}$,
$\bar{d} \cup \overline{av} = \overline{mv}$,	$\overline{dc} \cup \overline{nsc} = \overline{sc}$,	$\bar{a} \cup \overline{nsc} = \overline{ncv}$, and
$\bar{d} \cup \overline{nsc} = \overline{qsc}$,	$\overline{ns} \cup \bar{a} = \bar{n}$,	$\overline{av} \cup \overline{nsc} = \overline{nc}$.

$\bar{e}R_A$:  , $\overline{qs}R_A$:  , $\overline{nv}R_A$:  ,

$\bar{s}R_A$:  , $\overline{qm}R_A$:  , $\overline{nsca}R_A$:  ,

$\bar{m}R_A$:  , $\overline{qmv}R_A$:  , $\overline{csca}R_A$:  ,

$\overline{mv}R_A$:  , $\overline{sc}R_A$:  , $\overline{ncv}R_A$:  ,

$\overline{qsc}R_A$:  , $\bar{n}R_A$:  , $\overline{nc}R_A$:  .

2.2.4.3 <u>Unions of precisely three arguments</u>:

$\cup \{f_i : i \in K\}$, where $|K| = 3$.

There are $\binom{6}{3} = 20$ of those unions. Notice that $\{\bar{d}R_A, \overline{dc}R_A, \bar{a}R_A, \overline{av}R_A, \overline{ns}R_A, \overline{nsc}R_A\}$ is a partition of $(A \times A)_A$ for every R_A. Hence, whenever the union over K is known, the union over $\{1,\ldots,6\}$-K is just the complement of the former. It is sufficient to list the following ten and their complements:

$\bar{d} \cup \overline{dc} \cup \overline{ns} = \overline{rs}$, $\overline{crs} = \overline{ncs}$,

$\bar{d} \cup \overline{dc} \cup \bar{a} = \overline{ra}$, $\overline{cra} = \overline{nca}$,

$\bar{d} \cup \overline{dc} \cup \overline{av} = \overline{rav}$, $\overline{crav} = \overline{ncav}$,

$\bar{d} \cup \overline{dc} \cup \overline{nsc} = \overline{rsc}$, $\overline{crsc} = \overline{ncsc}$,

$\bar{d} \cup \overline{ns} \cup \bar{a} = \bar{i}$, $\bar{c}$,

$\bar{d} \cup \overline{ns} \cup \overline{av} = \bar{v}$, $\overline{cv}$,

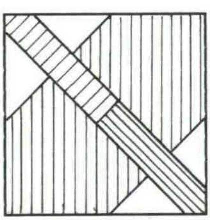
$\bar{d} \cup \overline{ns} \cup \overline{nsc} = \overline{qscm}$, $\overline{cqscm}$,

$\bar{d} \cup \bar{a} \cup \overline{av} = \overline{cscm}$, $\overline{scm}$,

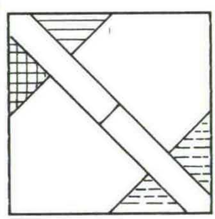
$\bar{d} \cup \bar{a} \cup \overline{nsc} = \overline{qcv}$, $\overline{cqcv}$, and

$\bar{d} \cup \overline{av} \cup \overline{nsc} = \overline{qc}$, $\overline{cqc}$.

$\overline{rs}R_A$:

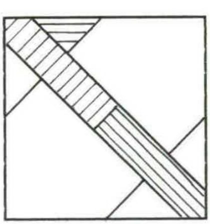


, $\overline{ncs}R_A$:

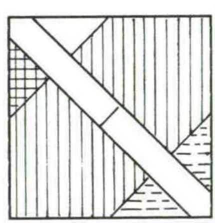


,

$\overline{ra}R_A$:

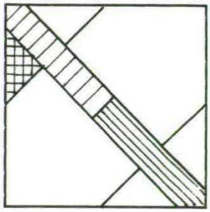


, $\overline{nca}R_A$:

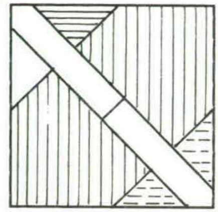


,

$\overline{\mathrm{rav}}R_A$:

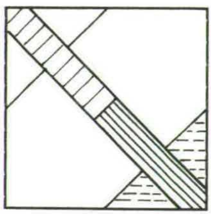


, $\overline{\mathrm{ncav}}R_A$:

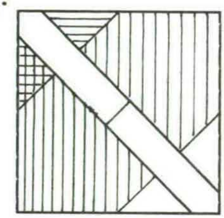


,

$\overline{\mathrm{rsc}}R_A$:

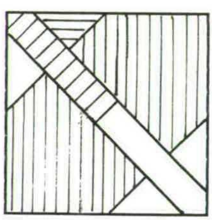


, $\overline{\mathrm{ncsc}}R_A$:

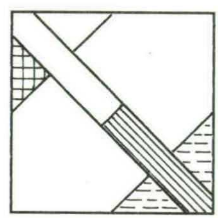


,

$\bar{\mathrm{i}}R_A$:

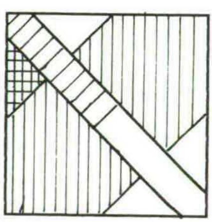


, $\bar{\mathrm{c}}R_A$:

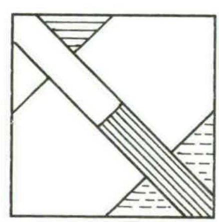


,

$\bar{\mathrm{v}}R_A$:

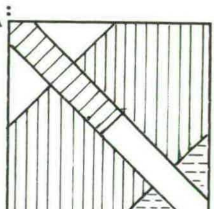


, $\overline{\mathrm{cv}}R_A$:

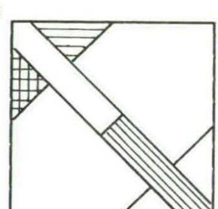


,

$\overline{\mathrm{qscm}}R_A$:

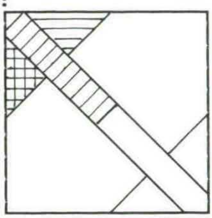


, $\overline{\mathrm{cqscm}}R_A$:

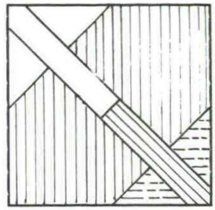


,

$\overline{cscm}R_A$:

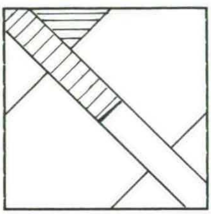


, $\overline{scm}R_A$:

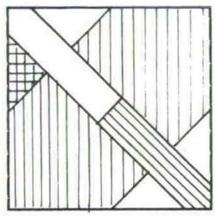


,

$\overline{qcv}R_A$:

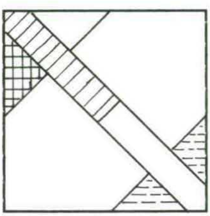


, $\overline{cqcv}R_A$:

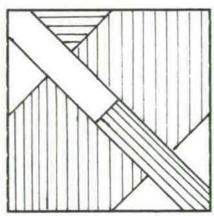


,

$\overline{qc}R_A$:



, $\overline{cqc}R_A$:



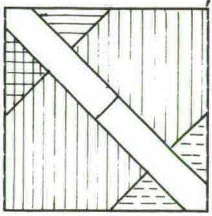
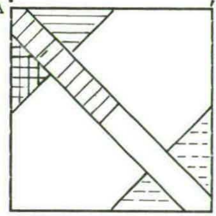
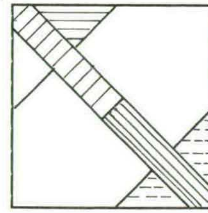
.

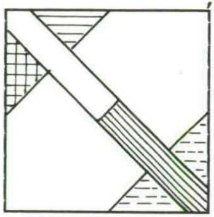
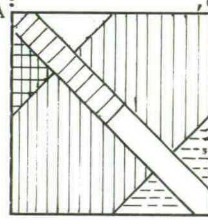
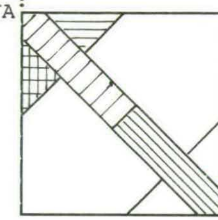
2.2.4.4 <u>Unions of precisely four arguments</u>:

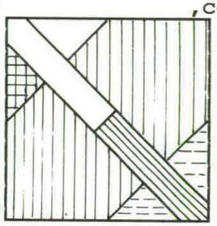
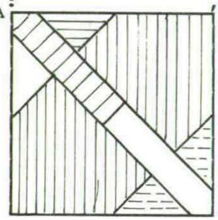
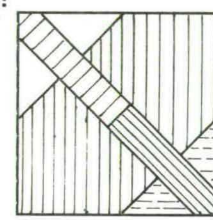
$\cup \{f_i : i \in K\}$, where $|K| = 4$.

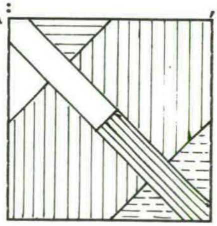
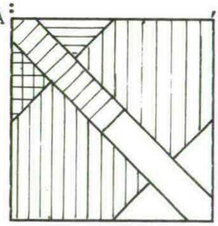
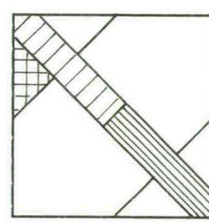
They are just the complements of the operations listed in (2.2.4.2):

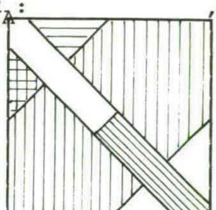
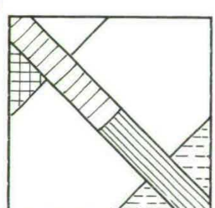
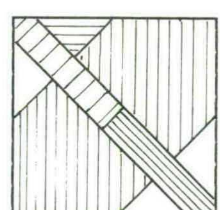
$\overline{ce}$,	$\overline{cqs}$,	$\overline{cnv}$,
$\overline{cs}$,	$\overline{cqm}$,	$\overline{cnsca}$,
$\overline{cm}$,	$\overline{cqmv}$,	$\overline{sca}$,
$\overline{cmv}$,	$\overline{csc}$,	$\overline{cncv} = \overline{rccv} = \overline{rv}$, and
$\overline{cqsc}$,	$\overline{cn}$,	$\overline{cnc} = \overline{rcc} = \bar{r}$.

$\overline{ce}R_A$: , $\overline{cqs}R_A$: , $\overline{cnv}R_A$: ,

$\overline{cs}R_A$: , $\overline{cqm}R_A$: , $\overline{cnsca}R_A$: ,

$\overline{cm}R_A$: , $\overline{cqmv}R_A$: , $\overline{sca}R_A$: ,

$\overline{cmv}R_A$: , $\overline{csc}R_A$: , $\overline{rv}R_A$: ,

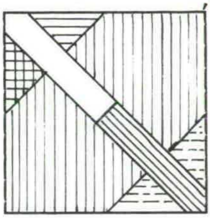
$\overline{cqsc}R_A$: , $\overline{cn}R_A$: , $\bar{r}R_A$: .

2.2.4.5 <u>Unions of precisely five arguments</u>:

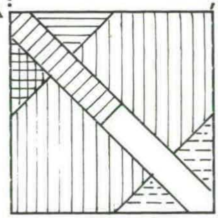
U $\{f_i: i \varepsilon K\}$, where $|K| = 5$.

They are just the complements of the operations listed in (2.2.4.1): $\overline{cd}$, $\overline{cdc}$, $\overline{cns}$, $\overline{ca}$, $\overline{cav}$ and $\overline{cnsc}$.

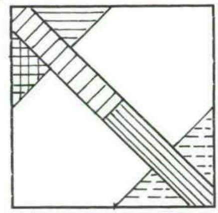
$\overline{cd}R_A$:



, $\overline{cdc}R_A$:

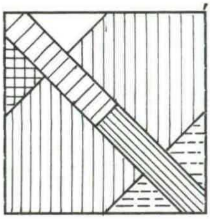


, $\overline{cns}R_A$:

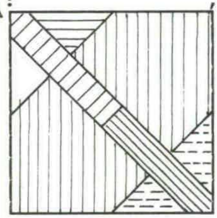


,

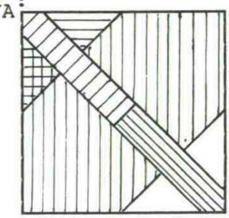
$\overline{ca}R_A$:



, $\overline{cav}R_A$:



, $\overline{cnsc}R_A$:



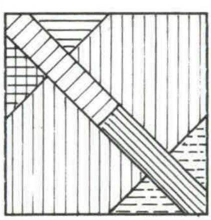
.

2.2.4.6 <u>Unions of precisely six arguments</u>:

U $\{f_i: i \varepsilon K\}$, where $|K| = 6$.

This is the complement of φ : $\overline{co}$.

$\overline{co}R_A$:



.

Finally, we like to answer the following two questions:

1. Which monadic operations based on local information do not lose information?

2. Which monadic operations based on local information do not yield images that conflict with its original?

2.2.4.7 <u>The answer to the first question</u>:

Note that a monadic operation h based on local information, which loses information, can be characterized as follows: There is a relation R_A ε Å such that R_A is the image of a relation in Å under h but we do not know which relation. Since h is based on local information, we can easily reconstruct possible originals for R_A. Hence, $h^{-1}(R_A)$ is not unique. So, h does not lose information precisely when there is an operation g based on local information such that $g = h^{-1}$ or, stated otherwise, h has an inverse operation based on local information. (Actually h is a permutation on Å).

It is straightforward to check that the following operations are the only bijective monadic operations based on local information: $\bar{i},\bar{v},\bar{c},\overline{cv},\bar{q},\overline{qv},\overline{qc}$ and $\overline{qcv}$.

These are the only operations h based on local information, such that for all R_A ε Å: $\{\bar{a}hR_A,\overline{av}hR_A\} = \{\bar{a}R_A,\overline{av}R_A\}$,

$\{\overline{ns}hR_A,\overline{nsc}hR_A\} = \{\overline{ns}R_A,\overline{nsc}R_A\}$, and

$\{\bar{d}hR_A,\overline{dc}hR_A\} = \{\bar{d}R_A,\overline{dc}R_A\}$.

By this property it is easy to construct R_A from $h(R_A)$ whenever h is known. Moreover, let M be the set of these operations: $\tilde{M} = \{\bar{i},\bar{v},\bar{c},\overline{cv},\bar{q},\overline{qv},\overline{qc},\overline{qcv}\}$; $\ulcorner\tilde{M},\circ\urcorner$ is an Abelian group, where every element is its own inverse with respect to composition. This is easily deduced from the "multiplication"-tabel of this group as shown below:

$\circ$	$\bar{i}$	$\bar{v}$	$\bar{c}$	$\bar{q}$	$\overline{qv}$	$\overline{cv}$	$\overline{qc}$	$\overline{qcv}$
$\bar{i}$	$\bar{i}$	$\bar{v}$	$\bar{c}$	$\bar{q}$	$\overline{qv}$	$\overline{cv}$	$\overline{qc}$	$\overline{qcv}$
$\bar{v}$	$\bar{v}$	$\bar{i}$	$\overline{cv}$	$\overline{qv}$	$\bar{q}$	$\bar{c}$	$\overline{qcv}$	$\overline{qc}$
$\bar{c}$	$\bar{c}$	$\overline{cv}$	$\bar{i}$	$\overline{qc}$	$\overline{qcv}$	$\bar{v}$	$\bar{q}$	$\overline{qv}$
$\bar{q}$	$\bar{q}$	$\overline{qv}$	$\overline{qc}$	$\bar{i}$	$\bar{v}$	$\overline{qcv}$	$\bar{c}$	$\overline{cv}$
$\overline{qv}$	$\overline{qv}$	$\bar{q}$	$\overline{qcv}$	$\bar{v}$	$\bar{i}$	$\overline{qc}$	$\overline{cv}$	$\bar{c}$
$\overline{cv}$	$\overline{cv}$	$\bar{c}$	$\bar{v}$	$\overline{qcv}$	$\overline{qc}$	$\bar{i}$	$\overline{qv}$	$\bar{q}$
$\overline{qc}$	$\overline{qc}$	$\overline{qcv}$	$\bar{q}$	$\bar{c}$	$\overline{cv}$	$\overline{qv}$	$\bar{i}$	$\bar{v}$
$\overline{qcv}$	$\overline{qcv}$	$\overline{qc}$	$\overline{qv}$	$\overline{qcv}$	$\bar{c}$	$\bar{q}$	$\bar{v}$	$\bar{i}$

2.2.4.8 The answer to the second question.

Let us illustrate the question with a few examples. There are relations $R_A \in \mathring{A}$ such that $\bar{v}R_A \cap \overline{av}R_A \neq \varphi$ or $\bar{c}R_A \cap \overline{av}R_A \neq \varphi$ or $\overline{ca}R_A \cap \overline{av}R_A \neq \varphi$. So $\bar{v}, \bar{c}$ and $\overline{ca}$ yield images which conflict with their originals.

We are looking for $h \in \hat{M}$, such that $h \cap \overline{av} = \bar{o}$. It is straightforward to prove that $\hat{N} = \{\bar{o}, \bar{d}, \overline{dc}, \overline{ns}, \bar{a}, \overline{nsc}, \bar{e}, \bar{s}, \bar{m}, \overline{qsc}, \overline{qs}, \overline{qm}, \overline{sc}, \bar{n}, \overline{nsca}, \overline{ncv}, \overline{rs}, \overline{ra}, \overline{rsc}, \bar{i}, \overline{qscm}, \overline{qcv}, \bar{q}, \overline{scm}, \overline{cv}, \overline{ncav}, \bar{r}, \overline{sca}, \overline{rcv}, \overline{cqmv}, \overline{cmv}, \overline{cav}\}$ is the set of all monadic operations which are based on local information and whose images do not conflict with their originals.

■

Now we have the following properties for these operations, which are easy to prove. Therefore the proof is left to the reader.

Proposition 2.2.5 Properties of operations

Suppose: $R_A, R_A^1, R_A^2, R_B^3 \in \mathring{A}$, $C \in Æ$, with $A \cap B = \varphi$ and $C \subseteq A$, and $\sigma \in S_A$. Then:

2.2.5.1 $\bar{v}(R_A^1 \cup R_A^2) = (\bar{v}R_A^1) \cup (\bar{v}R_A^2)$, $(\bar{v}R_A^1) \cap (\bar{v}R_A^2) = \bar{v}(R_A^1 \cap R_A^2)$,

$\bar{v}(R_A^1 \circ R_A^2) = (\bar{v}R_A^2) \circ (\bar{v}R_A^1)$, $\bar{v}[R_A]^k = [\bar{v}R_A]^k$,

$\bar{v}(R_A^1 \gg R_B^3) = (\bar{v}R_B^3) \gg (\bar{v}R_A^1)$, $\overline{vv}R_A = R_A$,

$\overline{vc}R_A = \overline{cv}R_A$, $\overline{vr}R_A = \overline{rv}R_A$,

$\bar{v}\sigma R_A = \sigma\bar{v}R_A$, $\overline{vs}R_A = \overline{sv}R_A = \bar{s}R_A$,

$\overline{va}R_A = \overline{av}R_A$, $\bar{v}(R_{A|_C}) = (\bar{v}R_A)|_C$,

$\overline{vt}R_A = \overline{tv}R_A$,

2.2.5.2 $\bar{c}(R_A^1 \cup R_A^2) = (\bar{c}R_A^1) \cap (\bar{c}R_A^2)$, $\bar{c}(R_A^1 \cap R_A^2) = (\bar{c}R_A^1) \cup (\bar{c}R_A^2)$,

$\bar{c}(R_A^1 \gg R_B^3) = (\bar{c}R_B^3) \gg (\bar{c}R_A^1)$, $\overline{cc}R_A = R_A$,

$\bar{c}\sigma R_A = \sigma\bar{c}R_A$, $\bar{c}(R_{A|_C}) = (\bar{c}R_A)|_C$, 2.2.5.3 $\sigma(R_A^1 \cup R_A^2) = \sigma R_A^1 \cup \sigma R_A^2$, $\sigma(R_A^1 \cap R_A^2) = (\sigma R_A^1) \cap (\sigma R_A^2)$,

$(\sigma R_A^1) \circ (\sigma R_A^2) = \sigma(R_A^1 \circ R_A^2)$, $\sigma[R_A]^k = [\sigma R_A]^k$,

$\sigma(R_A^1 \gg R_B^3) = (\sigma R_A^1) \gg (\sigma R_B^3)$, $\sigma\bar{r}R_A = \bar{r}\sigma R_A$,

$\sigma\bar{c}R_A = \bar{c}\sigma R_A$, $\sigma\bar{s}R_A = \bar{s}\sigma R_A$,

$\sigma\bar{a}R_A = \bar{a}\sigma R_A$, $\sigma\bar{t}R_A = \bar{t}\sigma R_A$,

$\sigma(R_{A|_C}) = (\sigma R_A)|_{\sigma(C)}$,

2.2.5.4 $(R_A^1 \cup R_A^2)|_C = (R^1_{A|_C}) \cup (R^2_{A|_C})$,

$(R_A^1 \cap R_A^2)|_C = (R^1_{A|_C}) \cap (R^2_{A|_C})$,

$(R^1_{A|_C}) \circ (R^2_{A|_C}) \subseteq (R_A^1 \circ R_A^2)|_C$,

$(\bar{r}R_A)|_C = \bar{r}(R_{A|_C})$,

2.2.5.5 $\bar{a}(R_A^1 \gg R_B^3) = (\bar{a}R_A^1) \gg (\bar{a}R_B^3)$, $\overline{aa}R_A = \bar{a}R_A$, $\overline{tt}R_A = \bar{t}R_A$,

$\overline{sa}R_A = \overline{as}R_A = \varphi_A$, $\overline{ss}R_A = \bar{s}R_A$, $\overline{tr}R_A = \overline{rt}R_A$, and

2.2.5.6 for all $m \in \hat{M}$: $m\bar{v}R_A = \bar{v}mR_A$ and $m(R_{A|_C}) = (mR_A)|_C$.

■

Recalling the notion of groupoid (See Rosenfeld [1968]), a groupoid is a pair $\ulcorner V,f \urcorner$, where V is a set and f a binary operation from V x V to V, (V is closed under the binary operation f) we can state the following theorem:

<u>Theorem 2.2.6</u>

2.2.6.1 $\ulcorner \hat{M},\circ \urcorner$ is a groupoid; it is even a semi group with identity.

2.2.6.2 $\ulcorner \hat{N}, \cup, \cap \urcorner$ is a bigroupoid.

2.2.6.3 $\ulcorner \hat{N},\circ \urcorner$ is a groupoid; it is even a semi group with identity.

Proof of Theorem 2.2.6

(2.2.6.1) Let $\bar{x},\bar{y} \in \hat{M}$.

It is sufficient to prove that $\bar{x} \circ \bar{y} \in \hat{M}$.

Note that $\bar{x} = f_1 \cup f_2 \cup \ldots \cup f_6$, where

$f_i \in F = \{\bar{d},\overline{dc},\bar{a},\overline{av},\overline{nsc},\overline{ns},\bar{o}\}$ for all $i \in \{1,\ldots,6\}$.

Since $\ulcorner \hat{M}, \cup, \cap \urcorner$ is a field, it is sufficient to prove that $f \circ \bar{y} \in \hat{M}$, for all $f \in F$. This is straightforward to prove.

(2.2.6.2) Let $\bar{x},\bar{y} \in \hat{N}$.

It is sufficient to prove that $\bar{x} \cup \bar{y} \in \hat{N}$ and $\bar{x} \cap \bar{y} \in \hat{N}$.

Now $\bar{z} \in \hat{N}$ iff $\bar{z}R_X \subseteq \overline{cav}R_X$, for all $R_X \in$ Å.

Hence, $\bar{x}R_X \subseteq \overline{cav}R_X$ and $\bar{y}R_X \subseteq \overline{cav}R_X$ for all $R_X \in$ Å, and so

$(\bar{x} \cup \bar{y})R_X \subseteq \overline{cav}R_X$ and $(\bar{x} \cap \bar{y})R_X \subseteq \overline{cav}R_X$ for all $R_X \in$ Å.

Hence, $\bar{x} \cup \bar{y}$, $\bar{x} \cap \bar{y} \in \hat{N}$.

(2.2.6.3) Let $\bar{x},\bar{y} \in \hat{N}$.

It is sufficient to prove that $\bar{x} \circ \bar{y} \in \hat{N}$.

Note that $\bar{x} = f_1' \cup f_2' \cup \ldots \cup f_5'$, where

$f_i' \in F' = \{\bar{d},\overline{dc},\bar{a},\overline{nsc},\overline{ns},\bar{o}\}$ for all $i \in \{1,\ldots,5\}$.

Since $\ulcorner \hat{N}, \cup, \cap \urcorner$ is a bigroupoid, it is sufficient to prove that $f' \circ \bar{y} \in \hat{N}$, for all $f' \in F'$. This is again straightforward to prove.

■

Later on the following theorem will be very useful.

Theorem 2.2.7

Suppose: R_X, $R_Y \in$ Å, a, $b \in Y$, x, $y \in X$ and $m \in \hat{M}$.

Furthermore, let $x \geq y : R_X$ iff $a \geq b : R_Y$,
$y \geq x : R_X$ iff $b \geq a : R_Y$, and
$x = y$ iff $a = b$.

Then $\langle x,y \rangle \in mR_X$ iff $\langle a,b \rangle \in mR_Y$.

Proof of theorem 2.2.7

Suppose: a,b,x,y,R_X,R_Y and m are as above.

Furthermore, suppose $\langle x,y \rangle \in mR_X$.

For reasons of symmetry it suffices to prove: $\langle a,b \rangle \in mR_Y$.

Since $m = f_1 \cup f_2 \cup f_3 \cup f_4 \cup f_5 \cup f_6$, where

$f_1, f_2, \ldots f_6 \in \{\bar{o}, \bar{a}, \overline{av}, \overline{ns}, \overline{nsc}, \bar{d}, \overline{dc}\}$ it is sufficient to prove

$\langle a,b\rangle \in mR_Y$ for the case that $m \in \{\bar{o}, \bar{a}, \overline{av}, \overline{ns}, \overline{nsc}, \bar{d}, \overline{dc}\}$. Now:

$\langle x,y\rangle \in \bar{a}R_X$ iff $\langle x,y\rangle \in R_X$ & $\langle y,x\rangle \notin R_X$ & $x \neq y$,

$\langle x,y\rangle \in \bar{a}R_X$ iff $\langle x,y\rangle \notin R_X$ & $\langle y,x\rangle \in R_X$ & $x \neq y$,

$\langle x,y\rangle \in \overline{ns}R_X$ iff $\langle x,y\rangle \in R_X$ & $\langle y,x\rangle \in R_X$ & $x \neq y$,

$\langle x,y\rangle \in \overline{nsc}R_X$ iff $\langle x,y\rangle \notin R_X$ & $\langle y,x\rangle \notin R_X$ & $x \neq y$,

$\langle x,y\rangle \in \bar{d}R_X$ iff $\langle x,y\rangle \in R_X$ & $\langle y,x\rangle \in R_X$ & $x = y$,

$\langle x,y\rangle \in \overline{dc}R_X$ iff $\langle x,y\rangle \notin R_X$ & $\langle y,x\rangle \notin R_X$ & $x = y$, and

$\langle x,y\rangle \in \bar{o}R_X$ iff "False".

Hence, $\langle a,b\rangle \in mR_Y$ follows evidently from the assumptions.

■

We continue by introducing some conditions for relations. Almost all the conditions stated here are well-known in literature, therefore we will not dwell upon them.

<u>Definition 2.2.8</u> <u>Restrictions for relations</u>

Let $R_A \in \AA$ be a relation on $A \in Æ$. Then R_A is:

2.2.8.1	<u>reflexive</u>	iff	$\bar{r}R_A = R_A$,
2.2.8.2	<u>transitive</u>	iff	$\bar{t}R_A = R_A$,
2.2.8.3	<u>asymmetric</u>	iff	$\bar{a}R_A = R_A$,
2.2.8.4	<u>symmetric</u>	iff	$\bar{s}R_A = R_A$,
2.2.8.5	<u>irreflexive</u>	iff	$R_A \cap Id_A = \varphi$,
2.2.8.6	<u>antisymmetric</u>	iff	$\bar{s}R_A \subseteq Id_A$,
2.2.8.7	<u>complete</u>	iff	$\bar{r}(R_A \cup \bar{v}R_A) = \langle A \times A, A\rangle$,
2.2.8.8	<u>strongly complete</u>	iff	$R_A \cup \bar{v}R_A = \langle A \times A, A\rangle$,
2.2.8.9	<u>connected</u>	iff	$\bar{t}(\bar{r}(R_A \cup \bar{v}R_A)) = \langle A \times A, A\rangle$,
2.2.8.10	<u>negative transitive</u>	iff	$\bar{t}(\bar{c}(R_A)) = \bar{c}R_A$,
2.2.8.11	<u>acyclic</u>	iff	$\bar{t}(\bar{a}(R_A))$ is asymmetric,
2.2.8.12	<u>quasi-transitive</u>	iff	$\bar{t}(\bar{a}(R_A)) \subseteq \bar{a}R_A$,
2.2.8.13	<u>P^t-transitive</u>	iff	$[\bar{a}R_A]^t \subseteq \bar{a}R_A$ $(t \in N,\ t \geq 1)$,
2.2.8.14	<u>$P^tI^mP^t$-transitive</u>	iff	$[\bar{a}R_A]^t \circ [\bar{s}R_A]^m \circ [\bar{a}R_A]^k \subseteq \bar{a}R_A$ $(t \geq 1,\ m \geq 0, k \geq 0)$.

■

(2.2.8.12) to (2.2.8.14) are just weakenings of the transitivity condition (2.2.8.2) (See Blau[1979], Blair & Pollack[1979], and Sen[1970]).

Later on we will state some other weakenings of (2.2.8.2).

Let us recall some well-known properties of these restrictions.

<u>Proposition 2.2.9</u>

Let $A \in Æ$, let $R_A \in Å$ be a relation on A.

Then the following holds:

2.2.9.1 R_A is transitive iff $\bar{t}R_A \subseteq R_A$ iff $R_A \circ R_A \subseteq R_A$,

2.2.9.2 R_A is negative transitive iff $\bar{c}R_A$ is transitive,

2.2.9.3 $\overline{cav}R_A$ is strongly complete, and

2.2.9.4 R_A is complete iff $\bar{c}R_A$ is antisymmetric.

■

The aim of this chapter is to classify special types of relations, namely preference orderings. We are looking for criteria on the basis of which one can say whether or not a given relation is a preference ordering. These criteria will be described as conditions on a set of relations. Hence, the question whether or not a given relation is a preference ordering, is translated to the problem whether or not the given relation is an element of a set of relations satisfying the proposed criteria. We will now introduce those criteria and explain how they are related to sets of preference orderings.

We will now state the first criterion:

<u>Definition 2.2.10</u> <u>Closed under permutation</u> <u>(Criterion 1)</u>

Let $\varphi \neq V \subseteq Å$.

V is <u>closed under permutation</u>, iff $\sigma R_X \in V$, for all $\sigma \in S_U$ and all $R_X \in V$.

■

A set V of relations is closed under permutation, iff for all relations R_X in V it holds that for every set of elements Y in Æ, such that $|X| = |Y|$, and for every rôle-exchange σ in S_U, such that $\sigma(X) = Y$, there is a relation R'_Y in V, such that the rôle of an arbitrary x in X played in R_X is played by $\sigma(x)$ in R'_Y.

Of course $R'_Y = \sigma R_X$. If V is closed under permutation and R_X is in V, then for every Y in Æ, such that $|Y| = |X|$, it holds that the elements in Y can play the rôle of the elements in X. So if V is closed under permutation it cannot give a discriminative treatment to a specific subset X ε Æ. For this reason we shall demand later on that the classified sets of orderings must be closed under permutation.

Before stating the second condition we discuss an invariance property of sets of relations which are closed under permutation.

<u>Proposition 2.2.11</u>

Let $\varphi \neq V \subseteq$ Å. Then it holds:

V is closed under permatation, iff $V = V^{\sigma}$, for all $\sigma \in S_U$, where $V^h := \{h(R_Y) : R_Y \in V\}$ for all monadic operations h.

<u>Proof of proposition 2.2.11</u>

Note that $R_X = \sigma\sigma^{-1}R_X$, by which (2.2.11) evidently holds.

■

The second criterion is as follows:

<u>Definition 2.2.12</u> <u>Closed under conversion</u> (<u>Criterion 2</u>)

Let $\varphi \neq V \subseteq$ Å.

V is <u>closed under conversion</u>, iff $\bar{v}R_X \in V$, for all $R_X \in V$.

■

A set of relations W is closed under conversion, iff for every relation R_X in W it holds that reversing all the preferences between the elements in R_X results in a new relation $\bar{v}R_X$ which is again in W. Criterion 1 and 2 are independent from each other, since there does not always exist a permutation σ such that for a relation $R_X \in$ Å, $\sigma R_X = \bar{v}R_X$.

Criterion 2 can also be interpreted as an invariance property, which is shown by the following proposition:

<u>Proposition 2.2.13</u>

Let $\varphi \neq V \subseteq$ Å. Then it holds:

V is closed under conversion, iff $V = V^{\bar{v}}$.

<u>Proof of proposition 2.2.13</u>

Note that $R_X = \bar{v}\bar{v}R_X$, by which (2.2.13) evidently holds.

■

The third criterion is as follows:

Definition 2.2.14 Closed under restriction (Criterion 3)

Let $\varphi \neq V \subseteq Å$.

V is closed under restriction, iff for all $R_A \in V$ and all $B \in Æ$, with $B \subseteq A$, $R_{A|_B} \in V$.

■

A set of relations W is closed under restriction, iff for every relation R_X in W it holds that every (complete) part of R_X ($R_{X|_Y}$, $\varphi \neq Y \subseteq X$) is an element of W. Alternatively, stated in terms ofpreference ordenings: W, a set of preference orderings, is closed under restriction, iff every relation R_X only consists of "subrelations" $R_{X|_Y}$ ($Y \subseteq X$, $\varphi \neq Y$), which are preference orderings in W. This means that the local ordering of a subset Y of X does not depend on the cardinality of Y or, stated otherwise, that the elements of X - Y are irrelevant with respect to order the elements of Y.

We have come to the fourth criterion:

Definition 2.2.15 Closed under concatenation (Criterion 4)

Let $\varphi \neq V \subseteq Å$.

V is closed under concatenation, iff for all $R_A^1, R_B^2 \in V$, with $A \cap B = \varphi$, $R_A^1 \gg R_B^2 \in V$.

■

A set of relations W is closed under concatenation, iff the concatenation of two relations R_A^1 and R_B^2, with $A \cap B = \varphi$, in W is again in W. In terms of preference orderings this means that having two orderings R_A^1 and R_B^2, such that $A \cap B = \varphi$, in W $R_A^1 \gg R_B^2$, which is built by just preferring everything in A to B and leaving R_A^1 and R_B^2 in $R_A^1 \gg R_B^2$ unchanged, is again in W. By this criterion we exclude sets of relations without any preference. This is easy to understand, since we create the possibility to prefer strictly by concatenation. Hence, by criterion 4 we demand some "ordering-principle" for a set $V \subseteq \AA$.

We introduce the fifth criterion:

<u>Definition 2.2.16</u> <u>Non-triviality</u>

Let $\varphi \neq V \subseteq \AA$.

V is <u>non-trivial</u>, iff for all $X \in \AE$ there are $R^1_X, R^2_X \in \AA$, such that $R^1_X \in V$ and $R^2_X \notin V$.

■

A set of relations W is non-trivial, iff W does not contain all possible relations on a set X in Æ and, furthermore, W does contain at least one relation on that set X. Hence, W restricted to X is not a trivial set of relations on X. This criterion could be omitted, but for simplicity we will impose it on a set of orderings.

Now we have come to the last criterion. Before stating it we introduce a substitution operation.

<u>Definition 2.2.17</u> <u>Substitution</u>

Let $R^1_X, R^2_Y \in \AA$, such that $X \cap Y = \varphi$, $x \in X$ and $\bar{v}R^2_Y = R^2_Y$.

Let $Z = (X \cup Y) - \{x\}$. Then:

$$\begin{aligned} \mathrm{Sub}\,(R^1_X, x, R^2_Y) := <\{<a,b>:\ & [a,b \in X-\{x\}\ \&\ <a,b> \in R^1_X]\ \vee \\ & [a,b \in Y\ \&\ <a,b> \in R^2_Y]\ \vee \\ & [a \in Y, b \in X-\{x\}\ \&\ <x,b> \in R^1_X]\ \vee \\ & [a \in X-\{x\}, b \in Y\ \&\ <a,x> \in R^1_X]\}, Z>. \end{aligned}$$

■

Of course $\mathrm{Sub}(R^1_X, x, R^2_Y) \in \AA$. When R^2_Y is reversible ($\bar{v}R^2_Y = R^2_Y$), then we can substitute R^2_Y in R^1_X, where R^2_Y plays the rôle of x in X. The following example explains the name of this operation.

<u>Example 2.2.18</u>

Suppose: $X = \{x,y,z\}$, $Y = \{a,b\}$, with $Y \cap X = \varphi$

$(\binom{(xz)}{y}) : R^1_X$ and $\binom{a}{b} : R^2_Y$. R^1_X: x.——.z and R^2_Y: .a
.y .b

Then $\mathrm{Sub}(R^1_X, x, R^2_Y)$ has the following representation:

$\left[\binom{(\binom{a}{b}z)}{y}\right] : \mathrm{Sub}(R^1_X, x, R^2_Y)$. $\mathrm{Sub}(R^1_X, x, R^2_Y)$: a.——.z——.b
.y

■

We state now the last criterion.

Definition 2.2.19 Closed under Substitution (Criterion 6)

Let $\varphi \neq V \subseteq \mathring{A}$.

V is closed under substitution, iff $\mathrm{Sub}(R^1_X, x, R^2_Y) \in V$, for all R^1_X, $R^2_Y \in V$, with $\bar{v}R^2_Y = R^2_Y$ and $X \cap Y = \varphi$, and all $x \in X$. ∎

A set of relations W is closed under substitution iff, for all relations R^1_X in W and elements x in X and for all reversible relations R^2_Y in W (i.e., $R^2_Y = \bar{v}R^2_Y$) which are completely disjoint from R^1_X (i.e., $X \cap Y = \varphi$), there is a relation R^3_Z in W in which R^2_Y plays the rôle of x in R^1_X. Of course $R^3_Z = \mathrm{Sub}(R^1_X, x, R^2_Y)$. Stated otherwise, a reversible relation can always be substituted in any preference ordering. The reversibility condition on R^2_Y is essential since otherwise the rôle of the reversible "x" has been taken over by an irreversible R^2_Y in R^1_X, which easily leads to intransitivities. On the other hand, if we drop this criterion the number of elements in a reversible part of an ordering becomes relevant. This is excluded by criterion 6.

Notation 2.2.20 Some standard set of relations

The following sets are standard for the rest of this monograph:

2.2.20.1 $Y_1 := \{\bar{c}\varphi_X \in \mathring{A} : |X| = 1\}$ is the set of reflexive relations on singletons,

2.2.20.2 $Y_2 := \{\varphi_X \in \mathring{A} : |X| = 1\}$ is the set of irreflexive relations on singletons,

2.2.20.3 $Y_3 := \{\bar{c}\varphi_X \in \mathring{A} : X \in Æ\}$ is the set of reflexive total indifference relations,

2.2.20.4 $Y_4 := \{\bar{n}\bar{c}\varphi_X \in \mathring{A} : X \in Æ\}$ is the set of irreflexive total indifference relations,

2.2.20.5 $Y_5 := \{\bar{r}\varphi_X \in \mathring{A} : X \in Æ\}$ is the set of reflexive total incomparable relations,

2.2.20.6 $Y_6 := \{\varphi_X \in \mathring{A} : X \in Æ\}$ is the set of irreflexive total incomparable relations,

2.2.20.7 $Y_7 := \{\bar{r}R_X \in \mathring{A} : \bar{v}R_X = R_X\}$ is the set of reflexive reversible relations, and

2.2.20.8 $Y_8 := \{\bar{n}R_X \in Å : \bar{v}R_X = R_X\}$ is the set of <u>irreflexive reversible relations</u>.

■

Clearly $Y_1 = \overline{Y_2^r}$, $Y_3 = \overline{Y_4^r}$, $Y_5 = \overline{Y_6^r}$, $Y_7 = \overline{Y_8^r}$, $\overline{Y_8^{cav}} = Y_3$ and $\overline{Y_8^{av}} = Y_6$.

Furthermore it is clear that, if $W \subseteq Å$ is closed under restriction, substitution and permutation then:

- $W \cap Y_i \neq \varphi$ iff $Y_i \subseteq W$ for $i \in \{1,2\}$,
- $W \cap (Y_i - Y_j) \neq \varphi$ iff $Y_i \subseteq W$ for $\langle i,j\rangle \in \{\langle 3,1\rangle,\langle 5,1\rangle,\langle 4,2\rangle,\langle 6,2\rangle\}$,
- $Y_7 \subseteq W$ iff $Y_5 \subseteq W$ and $Y_3 \subseteq W$, and
- $Y_8 \subseteq W$ iff $Y_6 \subseteq W$ and $Y_4 \subseteq W$.

It is possible to define the substitution mechanism more subtle, that is only elements in Y_i, for precisely one $i \in \{1,2,3,4,5,6\}$, may be substituted instead of every reversible relation. In doing so the theory remains essentially the same except two things:

(1) Some sets of relations often referred to as orderings will be classified as set of ordering with this more subtle substitution (Hence, this would clearly come closer to our intuitive notion of orderings),

(2) All the definitions, theorems and proofs become much more complicated.

Since the number of sets of relations that will not be classified as sets of orderings when using definition 2.2.17, and which are referred to by orderings, is small (to the best of our knowledge) and the degree of complexity in the other approach is great, we simplify the matter discussed in this chapter. Therefore (2.2.17) is used, although the theory using the more subtle substitution definition has already been developed (See earlier versions of this monograph).

Before defining which sets of relations are classified as sets of orderings, we dwell upon some properties of the substitution operation.

Proposition 2.2.21

Suppose: $R^1_X, R^2_X, R^1_Y, R^2_Y \in \mathring{A}$, with $X \cap Y = \varphi$, $R^1_Y = \bar{v}R^1_Y$ and $R^2_Y = \bar{v}R^2_Y$, and $x \in X$. Then:

2.2.21.1 $\mathrm{Sub}(R^1_X \cap R^2_X, x, R^1_Y \cap R^2_Y) = \mathrm{Sub}(R^1_X, x, R^1_Y) \cap \mathrm{Sub}(R^2_X, x, R^2_Y)$,

2.2.21.2 $\mathrm{Sub}(R^1_X \cup R^2_X, x, R^1_Y \cup R^2_Y) = \mathrm{Sub}(R^1_X, x, R^1_Y) \cup \mathrm{Sub}(R^2_X, x, R^2_Y)$,

2.2.21.3 $\bar{v}\mathrm{Sub}(R^1_X, x, R^1_Y) = \mathrm{Sub}(\bar{v}R^1_X, x, \bar{v}R^1_Y)$,

2.2.21.4 $\bar{c}\mathrm{Sub}(R^1_X, x, R^1_Y) = \mathrm{Sub}(\bar{c}R^1_X, x, \bar{c}R^1_Y)$,

2.2.21.5 $\bar{d}\mathrm{Sub}(R^1_X, x, R^1_Y) = \mathrm{Sub}(\bar{d}R^1_X, x, \bar{d}R^1_Y)$, and

2.2.21.6 for all $h \in \hat{M}$: $h\mathrm{Sub}(R^1_X, x, R^1_Y) = \mathrm{Sub}(hR^1_X, x, hR^1_Y)$.

Proof of proposition 2.2.21

(2.2.21.6) is an immediate result of (2.2.21.1) up to (2.2.21.5), since the elements of $\hat{M}$ can be built by $\cap$, $\cup$ and $\bar{c}$ from m_1, m_2 and m_3.

(2.2.21.1) This follows by the following observation:

$$
\begin{aligned}
&\langle a,b\rangle \in \mathrm{Sub}(R^1_X \cap R^2_X, x, R^1_Y \cap R^2_Y) && \leftrightarrow \\
&[\langle a,b\rangle \in R^1_X \ \&\ \langle a,b\rangle \in R^2_X \ \&\ a,b \in X - \{x\}] \ v \\
&[\langle a,b\rangle \in R^1_Y \ \&\ \langle a,b\rangle \in R^2_Y \ \&\ a,b \in Y\ ] \ v \\
&[\langle x,b\rangle \in R^1_X \ \&\ \langle x,b\rangle \in R^2_X \ \&\ b \in X - \{x\} \ \&\ a \in Y] \ v \\
&[\langle a,x\rangle \in R^1_X \ \&\ \langle a,x\rangle \in R^2_X \ \&\ a \in X - \{x\} \ \&\ b \in Y] && \leftrightarrow \\
\\
&\{[\langle a,b\rangle \in R^1_X \ \&\ a,b \in X - \{x\}] \ v \\
&[\langle a,b\rangle \in R^1_Y \ \&\ a,b \in Y\ ] \ v \\
&[\langle x,b\rangle \in R^1_X \ \&\ a \in Y \ \&\ b \in X - \{x\}] \ v \\
&[\langle a,x\rangle \in R^1_X \ \&\ a \in X - \{x\} \ \&\ b \in Y]\} \ \ \& \\
&\{[\langle a,b\rangle \in R^2_X \ \&\ a,b \in X - \{x\}] \ v \\
&[\langle a,b\rangle \in R^2_Y \ \&\ a,b \in Y\ ] \ v \\
&[\langle x,b\rangle \in R^2_X \ \&\ a \in Y \ \&\ b \in X - \{x\}] \ v \\
&[\langle a,x\rangle \in R^2_X \ \&\ a \in X - \{x\} \ \&\ b \in Y]\} && \leftrightarrow \\
&\langle a,b\rangle \in \mathrm{Sub}(R^1_X, x, R^1_Y) \cap \mathrm{Sub}(R^2_X, x, R^2_Y).
\end{aligned}
$$

(2.2.21.2) up to (2.2.21.5) are proved similarly.

■

We have come now to the main definition of this section.

<u>Definition 2.2.22</u> <u>Classifications of orderings</u>

Let $V \subseteq Å$.

V is <u>classified as a set of orderings</u>, iff

V is non-trivial and V is closed under permutation, conversion, restriction, concatenation and substitution.

■

In other words, V is classified as a set of orderings if V satisfies criteria 1 up to 6.

We end this section with some remarkable properties of sets which are classified as a set of orderings.

<u>Theorem 2.2.23</u>

Let $\varphi \neq V \subseteq Å$.

V is classified as a set of orderings iff (2.2.23.1), (2.2.23.2), and V is closed under conversion, restriction, concatenation and substitution, where

2.2.23.1 for all $A \in Æ$, with $|A| = 1$, there is a $R_A \in V$, and

2.2.23.2 either all $R_X \in V$ are reflexive or all $R_X \in V$ are irreflexive.

■

Theorem 2.2.23 states that a non-empty set of relations classified as a set of orderings either contains only reflexive relations or contains only irreflexive relations.

<u>Proof of theorem 2.2.23</u>

(only if) (2.2.23.1) follows immediately from the non-triviality of V.

It is sufficient to prove (2.2.23.2).

Suppose R_X, $R'_Y \in V$ such that there is a $x \in X$, with $\langle x,x\rangle \in R_X$ and there is a $y \in Y$, with $\langle y,y\rangle \notin R'_Y$; we deduce a contradiction and are done.

V is closed under restriction, so: $\langle\{\langle x,x\rangle\},\{x\}\rangle \in V$ and $\langle\varphi,\{y\}\rangle \in V$.

V is closed under permutation, so: $\{\{\langle x, x\rangle\}_{\{x\}}, \varphi_{\{x\}}\} \subseteq V$.

So, for $X = \{x\}$ the non-triviality of V is not satisfied. This contradicts our assumptions.

(if) Since permutations are in fact the result of a successive substitution of relations on singletons, the closedness under permutation follows from (2.2.23.1), (2.2.23.2) and the closedness under substitution of V.
The non-triviality is implied by (2.2.23.1), (2.2.23.2) and the closedness under concatenation of V.

■

Now the notion of order morphism is introduced.

<u>Definition 2.2.24</u> <u>Order morphism</u>

Let V and W be two sets of relations, such that V is classified as a set of orderings. A function h from V to W is an <u>order morphism</u>, iff:

2.2.24.1 $h(R_A) \in \{R'_B \in \mathring{A} : \bar{B} = A\}$ for all $R_A \in V$,

2.2.24.2 $h(\bar{v}R_A) = \bar{v}h(R_A)$ for all $R_A \in V$,

2.2.24.3 $h(\sigma R_A) = \sigma h(R_A)$ for all $R_A \in V$ and $\sigma \in S_U$,

2.2.24.4 $h(R_{A|_B}) = h(R_A)|_B$ for all $R_A \in V$ and $\varphi \neq B \subseteq A$,

2.2.24.5 $h(R^1_X \gg R^2_Y) = h(R^1_X) \gg h(R^2_Y)$ for all R^1_X, $R^2_Y \in V$, with $X \cap Y = \varphi$, and

2.2.24.6 $h(Sub(R^1_X,x,R^2_Y)) = Sub(h(R^1_X),x,h(R^2_Y))$ for all $x \in X$ and R^1_X, $R^2_Y \in V$, with $\bar{v}R^2_Y = R^2_Y$ and $X \cap Y = \varphi$.

Furthermore, h is an <u>order isomorphism</u>, iff h is a bijective order morphism. In that case V and W are <u>isomorph</u>.
Notation $V \cong W$.

■

In the following theorem we prove that the image of an order morphism is classifiable as a set of orderings, whenever the original space is classified as such. So, an order morphism preserves the introduced criteria, which explains the name order morphism.

<u>Theorem 2.2.25</u>

Let V and W be two sets of relations, such that V can be classified as a set of orderings. Suppose h is an order morphism from V to W. Then the set $h(V) := \{h(R_A) : R_A \in V\}$ is classified as a set of orderings.

Proof of theorem 2.2.25

Suppose V is classified as set of orderings and h is an order morphism. Let $R_A \in h(V)$ and $\sigma \in S_U$. So, there is a relation $R'_A \in V$, such that $h(R'_A) = R_A$.
Then $\sigma R_A = \sigma h(R'_A) = h(\sigma R'_A)$.
Since $\sigma R'_A \in V$, it follows $\sigma R_A \in h(V)$.
Hence, $h(V)$ is closed permutation.
Similarly it follows that $h(V)$ is closed under conversion, restriction, concatenation and substitution.
Let $X \in Æ$, $V(X) := \{R_A \in V : X = A\}$, and
$h(V)(X) := \{R_A \in h(V) : X = A\}$. $V(X)$ and $h(V)(X)$ are finite.
Now $h(V(X)) = h(V)(X)$, by which $h(V)$ is obviously non-trivial.

■

An investigation of the possible order morphisms between sets of orderings is the following point of interest. The next theorem is very useful in that investigation.

Theorem 2.2.26

Let $V, W \subseteq Å$, such that V can be classified as a set of orderings. Suppose h is an order morphism from V to W.

Then $h \in \{ \bar{n}, \bar{r}, \overline{ncav}, \overline{cav}, \overline{ncv}, \overline{rcv}, \bar{a}, \overline{ra}\}$.

Proof of theorem 2.2.26

The proof will be established in three steps.

Step 1 For all $a,b,c,d \in U$ and all $R_X, R_Y \in V$:

if $a \geq b : R_X$ iff $c \geq d : R_Y$, and
$b \geq a : R_X$ iff $d \geq c : R_Y$, and
$a = b$ iff $c = d$,
then $a \geq b : h(R_X)$ iff $c \geq d : h(R_Y)$.

Proof of step 1

Suppose: $a,b,c,d \in U$, $R_X, R_Y \in V$, $a \geq b : h(R_X)$,
$a = b$ iff $c = d$,
$a \geq b : R_X$ iff $c \geq d : R_Y$, and
$b \geq a : R_X$ iff $d \geq c : R_Y$.

We have to prove $c \geq d : h(R_X)$.

Let $\sigma \in S_U$ be such that $\sigma(a) = c$, $\sigma(b) = d$, $\sigma(c) = a$, $\sigma(d) = b$ and $\sigma(x) = x$ for $x \in U - \{a,b,c,d\}$.

Then obviously $R_X|_{\{a,b\}} = \sigma(R_Y|_{\{c,d\}})$.

Hence, we have the following sequence of implications:

$a \geq b : h(R_X)$ $\rightarrow$

$a \geq b : h(R_X)|_{\{a,b\}}$ $\rightarrow$

$a \geq b : h(R_X|_{\{a,b\}})$ $\rightarrow$

$a \geq b : h(\sigma(R_Y|_{\{c,d\}}))$ $\rightarrow$

$a \geq b : \sigma(h(R_Y)|_{\{c,d\}})$ $\rightarrow$

$\sigma(c) \geq \sigma(d) : \sigma(h(R_Y)|_{\{c,d\}})$ $\rightarrow$

$c \geq d : h(R_Y)$.

This completes the proof of step 1.

<u>Step 2</u> For all $R_X \in V : \bar{a}R_X = \bar{a}h(R_X)$.

<u>Proof of step 2</u>

Suppose $a > b : R_X$ for some $a,b \in X$ and $R_X \in V$.

Then $a > b : R_X \leftrightarrow a > b : R_X|_{\{a,b\}}$

$$\leftrightarrow R_X|_{\{a\}} \gg R_X|_{\{b\}} = R_X|_{\{a,b\}}$$

$$\rightarrow h(R_X)|_{\{a\}} \gg h(R_X)|_{\{b\}} = h(R_X)|_{\{a,b\}}$$

$$\leftrightarrow a > b : h(R_X).$$

Hence, $\bar{a}R_X \subseteq \bar{a}h(R_X)$ for all $R_X \in V$ (2.2.26.1)

Suppose not $a > b : R_X$.

We have to prove that not $a > b : h(R_X)$.

<u>Case 1</u> $b > a : R_X$.

By (2.2.26.1) $b > a : h(R_X)$.

Hence, not $a > b : h(R_X)$.

<u>Case 2</u> $\binom{a}{b} : R_X$ or $(ab) : R_X$.

Then we are ready by step 1.

This completes the proof of step 2.

Step 3 $\quad h \in \{\bar{n}, \bar{r}\ \overline{ncav}, \overline{cav}, \overline{ncv}, \overline{rcv}, \bar{a}, \overline{ra}\}$.

By step 1 and step 2 it follows:

(1) $\bar{a}R_X = \bar{a}h(R_X)$ and $\overline{av}R_X = \overline{av}h(R_X)$ for all $R_X \in V$,

(2a) either $(xy) : h(R_X)$ for all $R_X \in V$ and $x,y \in \overline{ns}R_X$

(2b) or $\binom{x}{y} : h(R_X)$ for all $R_X \in V$ and $x,y \in \overline{ns}R_X$,

(3a) either $(xy) : h(R_X)$ for all $R_X \in V$ and $x,y \in \overline{nsc}R_X$

(3b) or $\binom{x}{y} : h(R_X)$ for all $R_X \in V$ and $x,y \in \overline{nsc}R_X$, and

(4a) either $h(R_X)$ is reflexive for all $R_X \in V$

(4b) or $h(R_X)$ is irreflexive for all $R_X \in V$.

Now there are eight cases.

Case 1 (2a), (3a) and (4a).

For all $R_X \in V$ and all $x,y \in X$:

$x > y : h(R_X)$ iff $x > y : R_X$,

$(xy) : h(R_X)$ iff $(xy) : R_X$, $\binom{x}{y} : R_X$ or $x = y$.

Hence, $h = \overline{cav}$.

Case 2 (2a), (3a) and (4b). Leads similarly to $h = \overline{ncav}$.

Case 3 (2a), (3b) and (4a). Leads similarly to $h = \bar{r}$.

Case 4 (2a), (3b) and (4b). Leads similarly to $h = \bar{n}$.

Case 5 (2b), (3a) and (4a). Leads similarly to $h = \overline{rcv}$.

Case 6 (2b), (3a) and (4b). Leads similarly to $h = \overline{ncv}$.

Case 7 (2b), (3b) and (4a). Leads similarly to $h = \overline{ra}$.

Case 8 (2b), (3b) and (4b). Leads similarly to $h = \bar{a}$.

■

Note that for a classifiable set V of orderings, either all relations in V are reflexive or all are irreflexive, and that consequently:

$\bar{i} \in \{\bar{n},\bar{r}\}$, $\bar{q} \in \{\bar{n},\bar{r}\}$, $\overline{cv} \in \{\overline{ncv}, \overline{rcv}\}$, and $\overline{qcv} \in \{\overline{ncv}, \overline{rcv}\}$.

By combining several previous results the following corollary is deduced, where $V^h := \{hR_X : R_X \in V\}$ for all $V \subseteq Å$ and $h \in M$.

Corollary 2.2.27

Let $V,W \subseteq \AA$ such that V can be classified as a set of orderings.

Then the following holds:

2.2.27.1 for all $f \in \{\bar{n}, \bar{r}, \overline{ncav}, \overline{cav}, \overline{ncv}, \overline{rcv}, \bar{a}, \overline{ra}\}$, f is an order morphism from V to V^f,

2.2.27.2 for all $f \in \{\bar{i}, \bar{q}, \overline{cv}, \overline{qcv}\} = \tilde{M}$, f is an order isomorphism from V to V^f,

2.2.27.3 if f is an order morphism from V to $f(V) =: W$, then

$W \in \{V, V^{\overline{cv}}, V^{\bar{q}}, V^{\overline{qcv}}, V^{\bar{a}}, V^{\bar{m}}, V^{\overline{qm}}, V^{\bar{n}}, V^{\overline{ncv}}, V^{\overline{cav}}, V^{\overline{ra}},$

$V^{\overline{ncav}}, V^{\bar{r}}, V^{\overline{rcv}}, V^{\overline{cqmv}}, V^{\overline{cmv}}\}$ and W can be classified as a set of orderings, and

2.2.27.4 if W is order isomorph with V, then W is classified as set of orderings and $W \in \{V, V^{\overline{cv}}, V^{\bar{q}}, V^{\overline{qcv}}\}$.

Proof of corollary 2.2.27

(2.2.27.1) By (2.2.5) and (2.2.21) it follows that $f \in \{\bar{n}, \bar{r}, \overline{ncav}, \overline{cav}, \overline{ncv}, \overline{rcv}, \bar{a}, \overline{ra}\}$ is an order morphism.

(2.2.27.2) Notice that by theorem 2.2.23 for all $R_X \in V$, R_X is irreflexive or for all $R_X \in V$, R_X is reflexive.
Hence, $\bar{n}R_X = \bar{i}R_X$, for all $R_X \in V$, or $\bar{n}R_X = \bar{q}R_X$, for all $R_X \in V$, and $\bar{r}R_X = \bar{q}R_X$, for all $R_X \in V$, or $\bar{r}R_X = \bar{i}R_X$, for all $R_X \in V$, and $\overline{ncv}R_X = \overline{cv}R_X$, for all $R_X \in V$, or $\overline{ncv}R_X = \overline{qcv}R_X$, for all $R_X \in V$, and $\overline{rcv}R_X = \overline{qcv}R_X$, for all $R_X \in V$, or $\overline{rcv}R_X = \overline{cv}R_X$, for all $R_X \in V$.
Since $\{\bar{i}, \bar{q}, \overline{cv}, \overline{qcv}\} \subseteq \hat{M}$ we are ready.

(2.2.27.3) is a consequence of (2.2.26).

(2.2.27.4) $\bar{i}$, $\overline{cv}$, $\bar{q}$ and $\overline{qcv}$ are the only prder morphisms with an inverse.

■

We end this section with a theorem used later on.

Theorem 2.2.28

Let I be a collection of numbers and V_i for all $i \in I$ be a set of relations, which can be classified as a set of orderings. Suppose furthermore $W := \cap \{V_i : i \in I\} \neq \varphi$. Then W can be classified as a set of orderings.

Proof of theorem 2.2.28

Evident and therefore left to the reader.

■

§ 2.3 Linear Orderings.

In this section it is shown that the set of all linear orderings defined on the sets in Æ is a set of relations which can be classified as a set of orderings. Furthermore, we show that any set of relations $V \subseteq Å$, which can be classified as a set of orderings, has a subset $W \subseteq V$ which is isomorphic to the set of all linear orderings defined on the sets in Æ.

First let us dwell upon the name "linear ordering". In literature there are several names for an ordering which is reflexive, antisymmetric, complete and transitive. Some of these names are: "total ordering". (emphasizing the fact that the relation is strongly complete), "simple ordering", (emphasizing the fact that the relation is of a simple 'nature' because of its restrictive properties), "chain", (emphasizing the fact that the relation chains its elements from worse to better), or "linear ordering", (again emphasizing the chain-property of this ordering). The reader may verify these intuitive properties in, e.g., Roubens & Vincke [1985]. We will use the name linear ordering throughout.

In literature a linear ordering on a set X is a relation R_X such that R_X is reflexive, complete, antisymmetric and transitive. Hence, the set of linear orderings is

$L(U) := \{R_X \in Å$: R_X is reflexive, transitive, antisymmetric, and complete$\}$.

We will prove that L(U) can be classified as a set of orderings. In order to prove this, we introduce another way to describe L(U) and prove some general results which will be used in the following sections.

First, we introduce a mapping l from Å to Å, whose "nil-points" together form precisely the set of reflexive, complete, antisymmetric and transitive orderings. We use this function to describe L(U).

Let l: Å $\rightarrow$ Å be defined as follows:

$$l(R_X) := (\bar{r}R_X \cap \bar{c}R_X) \cup (\overline{cr}\,(R_X \cup \bar{v}R_X)) \cup (R_X \cap \bar{v}R_X \cap \bar{c}Id_X) \cup ((R_X \circ R_X) \cap \bar{c}R_X).$$

Now $l(R_X) = \varphi_X \iff \bar{r}R_X \cap \bar{c}R_X = \varphi_X$ & $\overline{cr}(R_X \cup \bar{v}R_X) = \varphi_X$ &
$(R_X \cap \bar{v}R_X) \cap \bar{c}Id_X = \varphi_X$ &
$R_X \circ R_X \cap \bar{c}R_X = \varphi_X$

$\iff \bar{r}R_X \subseteq R_X$ & $\bar{r}(R_X \cup \bar{v}R_X) = \langle X \times X, X\rangle$ &
$R_X \cap \bar{v}R_X \subseteq Id_X$ & $R_X \circ R_X \subseteq R_X$

$\iff R_X$ is reflexive, complete, antisymmetric and transitive.

Hence, $L(U) = \{ R_X \in Å: l(R_X) = \varphi_X\}$.

Let $v_1(R_X) := \bar{r}R_X \cap \bar{c}R_X$,

$v_2(R_X) := \bar{c}(\bar{r}(R_X \cup \bar{v}R_X))$,

$v_3(R_X) := (\bar{a}R_X \circ \bar{a}R_X) \cap \overline{ca}R_X))$ and

$v_4(R_X) := R_X \cap \bar{v}R_X \cap \bar{c}Id_X$.

Furthermore, let $V_1(U) = \{R_X \in Å: v_1(R_X) = \varphi_X\}$,

$V_2(U) = \{R_X \in Å: v_2(R_X) = \varphi_X\} \cap V_1(U)$,

$V_3(U) = \{R_X \in Å: v_3(R_X) = \varphi_X\} \cap V_1(U)$, and

$V_4(U) = \{R_X \in Å: v_4(R_X) = \varphi_X\} \cap V_1(U)$.

We will prove that $V_1(U), V_2(U), V_3(U)$ and $V_4(U)$ are sets of relations which can be classified as sets of orderings. As a corollary of this and theorem 2.2.28 it follows that $L(U)$ can be classified as a set of orderings, since $L(U) = V_1(U) \cap V_2(U) \cap V_3(U) \cap V_4(U)$.

<u>Lemma 2.3.1</u>

$V_1(U)$ can be classified as a set of orderings.

<u>Proof of lemma 2.3.1</u>

Evident, since $V_1(U)$ is the set of reflexive orderings.

■

<u>Lemma 2.3.2</u>

$V_2(U)$ can be classified as a set of orderings.

<u>Proof of lemma 2.3.2</u>

$V_2(U) = \{R_X \in Å: v_1(R_X) \cup v_2(R_X) = \varphi_X\}$.

$V_2(U) = \{R_X \in Å: R_X$ is reflexive and $v_2(R_X) = \varphi_X\}$.

$v_2(R_X) = \varphi_X$ iff R_X is complete.

So (2.2.23.1) and (2.2.23.2) hold for $V_2(U)$.

It is then sufficient to prove that $V_2(U)$ is closed under conversion, restriction, concatenation and substitution.
By (2.2.6) it follows that $v_2 \varepsilon M$.
Let $R_X \varepsilon V_2(U)$.
Then $v_2(R_X) = \varphi_X$ and R_X is reflexive.
Now we have by (2.2.5.6) $v_2\bar{v}(R_X) = \bar{v}v_2(R_X) = \bar{v}\varphi_X = \varphi_X$.
Clearly $\bar{v}R_X$ is reflexive.
Hence, $V_2(U)$ is closed under conversion.
Similarly it follows by (2.2.5.6) and (2.2.21.6) that $V_2(U)$ is closed under restriction and substitution.
Let R^1_X and R^2_Y be in $V_2(U)$, with $X \cap Y = \varphi$.
Since R^1_X and R^2_Y are reflexive and complete, it follows by the definition of » that R^1_X » R^2_Y is reflexive and complete.
Hence, $V_2(U)$ is closed under concatenation and by (2.2.23) it follows that $V_2(U)$ is classified as a set of orderings.

■

<u>Lemma 2.3.3</u>

$V_4(U)$ can be classified as a set of orderings.

<u>Proof of lemma 2.3.3</u>

By (2.2.27.3) and (2.3.2) it is sufficient to prove that

$V_2(U)^{\overline{rcv}} = V_4(U)$. Now for all $R_X \varepsilon V_1(U)$:

$R_X \varepsilon V_2(U)^{\overline{rcv}}$ iff

$\overline{rcv}R_X = \overline{cnv}R_X$ is reflexive and complete iff

$\overline{nv}R_X$ is irreflexive and antisymmetric (by (2.2.9.4)) iff

R_X is reflexive and antisymmetric iff

$R_X \varepsilon V_4(U)$.

By this the proof is obvious.

■

A logical following step would be to prove that $V_3(U)$ can be classified as a set of orderings. Instead of proving this, we prove a more general theorem, which is used in the next sections. We will fix our attention to a more general transitivity concept. In order to clarify to the reader that there are several transitivity conditions we remind him of definition 2.2.8.2, 2.2.8.13 and 2.2.8.14 and the following facts for a relation R_A in Å:

* if R_A is asymmetric, then R_A is P^t-transitive for all $t \geq 2$, whenever R_A is transitive, and
* if R_A is reflexive, then R_A is P^{t+k}-transitive, whenever R_A is $P^tI^mP^k$-transitive.

These facts show that in some 'non-trivial situations' (2.2.8.13) is a weakening of (2.2.8.14) and (2.2.8.13) is a weakening of (2.2.8.2). Therefore it is reasonable to call all of these conditions transitivity conditions. Moreover, in literature P^2-transitivity is often called quasi-transitivity. Obviously, we can weaken condition (2.2.8.14) even more by introducing $P^{k_1}I^{k_2}P^{k_3}P^{k_4}I^{k_5}P^{k_6}\ldots I^{k_s}P^{k_{s+1}}$-transitivity, which is defined similar to (2.2.8.14). Realizing that there are in fact infinitely many transitivity-conditions, the natural question arises whether there is a 'model' to describe these kinds of transitivity conditions.

Recall that a relation R_A is PIP-transitive, iff $\bar{a}R_A \circ \bar{s}R_A \circ \bar{a}R_A \subseteq \bar{a}R_A$ or, stated otherwise, iff for all $\langle x_0, x_1\rangle \in \bar{a}R_A$, $\langle x_1, x_2\rangle \in \bar{s}R_A$ and $\langle x_2, x_3\rangle \in \bar{a}R_A$, it holds that $\langle x_0, x_3\rangle \in \bar{a}R_A$.

Hence, this PIP-transitivity condition can be interpreted as follows:

A relation R_A is PIP-transitive, iff for all $x_0, x_1, x_2, x_3 \in A$ it holds that: if the preference of x_0 to x_1 in R_A can pass into an indifference between x_1 and x_2 in R_A and this indifference can pass into another preference of x_2 to x_3 then there exists a short cut of this transition, namely x_0 is preferred to x_3 in R_A.

In this interpretation we notice that if we call "prefer to" and "indifferent between" preference types of a relation, then the PIP-transitivity can be described as saying that special sequences of preference types of a relation can be shortened. This notion will lead to the general transitivity condition. In fact, we will formulate this notion in a more abstract sense to obtain the general transitivity condition.

A preference type of a relation is information about elements which are ordered according to that preference type. Hence, a preference type of a relation is in fact a <u>monadic</u>

operation on this relation. The next example illustrates what can happen with respect to transitivity conditions, whenever the monadic operation, which describes a preference type, is not based on local information. By this example we hope to convince the reader that the conditions deduced above are more or less natural and intuitively correct.

<u>Example 2.3.4</u>

Let $R_A \in \mathring{A}$, $B \subseteq A$, $B \in Æ$, $\sigma \in S_U$.

First we will study monadic operations which are based on information not known to the relation R_A. One might (intuitively) know, that operations on a "system" which are partly based on information not known to that "system" easily cause "discontinuities", "discrimination" or other odd properties. We will give two explicit illustrations of these facts.

Observe the following condition:

$$R_A \circ \sigma R_A \subseteq R_A. \qquad (2.3.4.1)$$

Suppose R_A fulfills condition (2.3.4.1).

Furthermore, let $\langle x,y\rangle \in R_A$, $\langle \sigma(y), z\rangle \in \sigma R_A$ and $\sigma(y) \neq y$.

(Hence, σ is not the identity).

Then it follows that $\langle x,z\rangle \in R_A$. But there is a discontinuity in the transition of preference types:

$x \geq y : R_A$ passes into $\sigma(y) \geq z: \sigma R_A$ and then is shortened to $x \geq z : R_A$, while $\sigma(y) \neq y$.

Observe the following condition:

$$R_A \circ R_{A|_B} \subseteq R_A \qquad (2.3.4.2)$$

It is evident that this condition discriminates between the elements in B and in A-B.

Next we concentrate on operations which are based only on information known to the relation R_A while this information is not local, hence global. Conditions based on this kind of operations can be very unstable in a special sense illustrated below.

Observe the following condition:

$$\overline{at}R_A \circ \overline{at}R_A \subseteq \bar{a}R_A. \qquad (2.3.4.3)$$

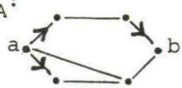
Consider the following relations represented by their graphs:

R_A^1:

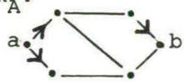
R_A^2:

R_A^3:

R_A^4:



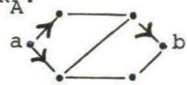
R_A^5:



R_A^6:

R_A^7:

R_A^8:



R_A^9:

R_A^{10}:

Then R_A^1 and R_A^{10} do not satisfy condition (2.3.4.3), while all the others do.
Note that the relations are very similar, hence (2.3.4.3) is an unstable conditon.

■

In order to avoid oddities as discussed in example 2.3.4 it is necessary to describe preference types in a transition sequence of transitivity conditions by monadic operations based on local information, that is, by operators in $\hat{M}$ (See example 2.2.4). Now it is clear by the discussion at (2.2.4.8) that only the monadic operations in $\hat{N}$ have images which do not conflict with their originals. Therefore, preference types of a relation are described by operations in $\hat{N}$ (See 2.2.4.8).

Next we formalize the above discussed intuitive notions.

Definition 2.3.5

Let $R_A \in \mathring{A}$, $C = \{x_0,\dots,x_k\}$, $C \subseteq B \subseteq A$ and $f_1,f_2,\dots,f_k \in \hat{N}$.

2.3.5.1 - A word over the alphabet $\hat{N}$ is the concatenation of zero or more symbols of $\hat{N}$
(See also Lewis & Papadimitriou [1981])

- If $w = f_1f_2\dots f_k$ we say that w has length k, f_i is the i^{th} symbol of w and $w^{\bar{v}} = f_k\dots f_1$ is the conversed word of w.
- $\hat{N}^+ := \{w : w$ is a word over $\hat{N}$ of positive length$\}$.
(Whenever a $f \in \hat{N}$ is the composition of $\bar{g}_1,\bar{g}_2,\dots,\bar{g}_t \in \hat{N}$ this composition will be denoted by a joint bar over the symbol in w: $f = \overline{g_1g_2g_3\dots g_t}$).

2.3.5.2 - Suppose: $w^1,w^2 \in \hat{N}^+$, $w^1 = f_1f_2\dots f_k$, and $w^2 = g_1g_2\dots g_m$.
w^1 is embedded in w^2, iff there is a function $h : \{1,\dots,k\} \to \{1,\dots,m\}$ such that:
$h(i) < h(j)$ for all $1 \le i < j \le k$ (h is monotonic), and
$f_i = g_{h(i)}$ for all $1 \le i \le k$.
- h is called an embedding.

2.3.5.3 - $\pi = \langle x_0,x_1,\dots,x_k\rangle$ is a path (from x_0 to x_k) (along R_A) (in B) (of type $w = f_1f_2\dots f_k$) iff
$\langle x_t, x_{t+1}\rangle \in f_{t+1}R_A$ for all $0 \le t \le k-1$.
- The length of π is k.
- Path π is a cycle iff $x_0 = x_k$.
- $\pi^{\bar{v}} = \langle x_k, x_{k-1},\dots,x_0\rangle$ is the conversed path of π (along $\bar{v}R_A$ of type $w^{\bar{v}}$).

2.3.5.4 - Let $\pi^1 = \langle x_0, x_1,\dots,x_k\rangle$ be a path along R_A of type w^2, $\pi^2 = \langle y_0, y_1,\dots,y_n\rangle$ a path along R_A and $w^1 \in \hat{N}^+$.
π^2 is a w^1-short cut of π^1 (along R_A) iff
π^2 is of type w^1,
w^1 is embedded in w^2, and
$\{y_0,\dots,y_n\} \subseteq \{x_0,\dots,x_k\}$ and $y_0 = x_0$, $x_k = y_n$. ∎

Example 2.3.6

Suppose: $w^1 = \overline{aaaiiiieeeeeaeeeeeeeaaaaaaaaaaaaaa}$

$= \bar{a}^3\bar{i}^4\bar{e}^5\bar{a}^1\bar{e}^7\bar{a}^{14}$ (abbreviated form of w^1),

$w^2 = \bar{a}^{18}$,

$w^3 = \bar{a}^3\bar{e}^{12}$,

$w^4 = \bar{a}^3\bar{e}^5\bar{a}^2\bar{e}^3$,

$w^5 = \overline{asa}$, and

$w^6 = \bar{a}$.

\+ Suppose: $R_A \in \AA$ $\langle x_0, x_1\rangle \in \bar{a}R_A$, $\langle x_1, x_2\rangle \in \bar{s}R_A$, $\langle x_2, x_3\rangle \in \bar{a}R_A$ and $\langle x_0, x_3\rangle \in \bar{a}R_A$.
Then $\langle x_0, x_1, x_2, x_3\rangle$ is a path from x_0 to x_3 along R_A of type w^5.

\+ Of course w^6 is embedded in w^5 (h(1):=1), hence $\langle x_0, x_3\rangle$ is a w^6-short-cut of $\langle x_0, x_1, x_2, x_3\rangle$.

\+ Note that the length of w^1 is 34, the 9^{th}-symbol of w^4 is $\bar{a}$, $(w^2)^{\bar{v}} = w^2$ and $(w^3)^{\bar{v}} = \bar{e}^{12}\bar{a}^3$.

\+ w^2 is embedded in w^1.
We only define the embedding.
h: $\{1,...,18\} \to \{1,...,34\}$

$$h(t) := \begin{cases} t & :\Leftrightarrow 1 \le t \le 3 \\ t+g & :\Leftrightarrow t = 4 \\ t+16 & :\Leftrightarrow 5 \le t \le 18 \end{cases}$$

Clearly h is an embedding.

\+ w^3 is embedded in w^1.
Again we only define an embedding.
g : $\{1,...,15\} \to \{1,...,15\}$.

$$g(t) := \begin{cases} t & :\Leftrightarrow 1 \le t \le 3 \\ t+4 & :\Leftrightarrow 4 \le t \le 8 \\ t+5 & :\Leftrightarrow 9 \le t \le 15 \end{cases}$$

\+ It is evident that w^4 is not embedded in w^1, since one can not find a subsequence of w^4 in w^1.

■

Intuitively, a word w_2 is embedded in w_1, whenever we can find a subsequence of symbols in w_1, with the same ordering of the symbol occurences as in w_1, which is equal to w_2. An embedding h preserves the ordering of the symbol occurences and the equality to w_1 is guaranteed by the second condition of (2.3.5.2).

Let us make some remarks about definition 2.3.5:

+ Let $\pi = \langle x_0,..,x_k\rangle$ be a path along R_A of type $w = f_1 f_2 ... f_k$, then w is a transition sequence of preference types occurring in π. So $\langle x_{i-1},x_i\rangle$ has preference type f_i, i.e., $\langle x_{i-1},x_i\rangle \in f_i R_A$ for all $i \in \{1,2,....k\}$.

+ Of course a type of a path is not unique. For instance, let $w_1 = \bar{i}\bar{a}\bar{e}\overline{cav}$ $w_2 = \overline{cav}^4$ and $w_3 = \bar{i}^2\bar{e}\ \overline{cav}$ and π a path along R_A. If π is of type w_1, then π is also of type w_2 and w_3 because $\bar{i}R_A \subseteq \overline{cav}R_A$, $\bar{a}R_A \subseteq \bar{i}R_A$ and $\bar{e}R_A \subseteq \overline{cav}R_A$.

The following lemma is used later on and shows some relationship between paths and their types.

<u>Lemma 2.3.7</u>

Let $\pi = \langle x_0,x_1,...,x_k\rangle$ be a path along R_A ($\in$ Å) of type $w \in \hat{N}^+$, w_1,w_2 two words in $\hat{N}^+$ and $\sigma \in S_U$.

Then the following holds:

2.3.7.1 $\pi^{\bar{v}}$ is of type $w^{\bar{v}}$,

2.3.7.2 $\pi^{\sigma} = \langle \sigma x_0,\sigma x_1,...,\sigma x_k\rangle$ is a path along σR_A of type w, and

2.3.7.3 If w_1 is embedded in w_2, then $w_1^{\bar{v}}$ is embedded in $w_2^{\bar{v}}$.

<u>Proof of lemma 2.3.7</u>

Let $\pi = \langle x_0,x_1,...,x_k\rangle$ be a path along R_A of type $w = f_1 f_2 ... f_k$.

Furthermore, let $w_1 = w$ and $w_2 = g_1 g_2 ... g_n$ and $\sigma \in S_U$.

(2.3.7.1) $\pi^{\bar{v}} = \langle x_k, x_{k-1},...x_1\rangle$ is a path along $\bar{v}R_A$. Furthermore, it is evident by (2.2.5) that $\langle x_t,x_{t-1}\rangle \in \bar{v}f_t R_A = f_t\bar{v}R_A$, for all $k \ge t \ge 1$.

Hence, $\pi^{\bar{v}}$ is a path along $\bar{v}R_A$ of type $w^{\bar{v}}$.

(2.3.7.2) $\pi^{\sigma} = \langle \sigma x_0, \sigma x_1,...,\sigma x_k\rangle$ is a path along σR_A. Note that for all $0 \le t \le k-1$ it holds that:

$\langle x_t, x_{t-1}\rangle \in f_tR_A$ iff $\langle \sigma x_t, \sigma x_{t-1}\rangle \in \sigma f_tR_A$

iff $\langle \sigma x_t, \sigma x_{t-1}\rangle \in f_t\sigma R_A$.

Hence, π^σ is of type w.

(2.3.7.3) Suppose w_1 is embedded in w_2, so there is a h from $\{1...k\}$ to $\{1...n\}$, such that:

$h(i) < h(j)$ for all $1 \le i < j \le k$, and

$f_i = g_{h(i)}$ for all $1 \le i \le k$.

Let $w_1^{\bar{v}} = f_kf_{k-1}...f_1 =: f_1'f_2'...f_k'$ and

$w_2^{\bar{v}} = g_n...g_1 =: g_1'g_2'...g_n'$.

Take h' from $\{1,...,k\}$ to $\{1,...,n\}$ such that $h'(i) = n-h(k-i+1)+1$.

If $1 \le i < j \le k$, then $(k-j+1) < (k-i+1)$.

So $h(k-j+1) < h(k-i+1)$ and

$h'(i) = n-h(k-i+1)+1 < h'(j) = n-h(k-j+1)+1$.

h' is monotonic.

Furthermore, for all $1 \le i \le k$:

$f_i' = f_{k-i+1} = g_{h(k-i+1)} = g'_{n-h(k-i+1)+1} = g'_{h'(i)}$.

■

Remember that $\tilde{M} = \{\bar{i}, \bar{v}, \bar{c}, \overline{cv}, \bar{q}, \overline{qv}, \overline{qc}, \overline{qcv}\}$ (Definition 2.2.4.7) is the set of all monadic operations (on relations in Å), based on local information, which do not loose information.

Now we are able to define a general transitivity condition.

<u>Definition 2.3.8</u> <u>Transitivity</u>

Let $R_A \in$ Å, $w_1, w_2 \in \hat{N}^+$ and $m \in \tilde{M}$. Then R_A is $\langle w_1, w_2\rangle$-<u>transitive with respect to</u> m, iff for every path π along mR_A of type w_1 there is a w_3-short cut along mR_A such that w_3 can be embedded in w_2.

■

Let w_1, w_2 be two given words in $\hat{N}^+$. Furthermore, let $R_A \in$ Å and $m \in \tilde{M}$ a monadic operator based on local information, which does not lose information. Then R_A in Å is $\langle w_1, w_2\rangle$-transitive with repect to m, iff for every path π_1 from x_0 to x_k of preference transitions, described by the sequence of preference types in w_1, along the relation mR_A, which has the same information value as R_A, can be cut short by a path π_2 from x_0 to x_k of preference transitions, described by a subsequence of w_2, along mR_A, which preserves the ordering of the occurences of the preference types. Hence, R_A is $\langle w_1, w_2\rangle$-transitive based on m, iff all special sequences of preference transitions based on the information of mR_A and described by w_1, can be cut short by a subsequence of preference transitions based on the information of mR_A and described by w_2.

If R_A is $\langle w_1, w_2\rangle$-transitive with respect to $\bar{i}$, then we say that R_A is $\langle w_1, w_2\rangle$-transitive.

In the following example we will show that some of the well-known transitivity-conditions known from literature can be described as special cases of (2.3.8).

<u>Example 2.3.9</u>

Let $R_A \in \mathring{A}$ be a relation

+ <u>Transitive</u>

  R_A is transitive iff

  $R_A \circ R_A \subseteq R_A$ iff

  R_A is $\langle \bar{i}^2, \bar{i}\rangle$-transitive.

+ <u>Negatively transitive</u>

  R_A is negatively transitive iff

  $\bar{c}R_A \circ \bar{c}R_A \subseteq \bar{c}R_A$ iff

  R_A is $\langle \bar{i}^2, \bar{i}\rangle$-transitive with respect to $\bar{c}$.

+ <u>Quasi transitive</u>

  R_A is quasi-transitive iff

  $\bar{a}R_A \circ \bar{a}R_A \subseteq \bar{a}R_A$ iff

  R_A is $\langle \bar{a}^2, \bar{a}\rangle$-transitive.

+ <u>$P^tI^mP^k$-transitive</u>

  R_A is $P^tI^mP^k$-transitive iff

  $[\bar{a}R_A]^t \circ [\bar{s}R_A]^m \circ [\bar{a}R_A]^k \subseteq \bar{a}R_A$ iff

  R_A is $\langle \bar{a}^t\bar{s}^m\bar{a}^k, \bar{a}\rangle$-transitive.

+ <u>Acyclic</u>

R_A is acyclic iff

for all $t \geq 1$: $[\bar{a}R_A]^t \cap \overline{av}R_A = \varphi$ iff

for all $t \geq 1$: $[\bar{a}R_A]^t \subseteq \overline{cav}R_A$ iff

R_A is $\langle \bar{a}^t, \overline{cav}\rangle$-transitive for all $t \geq 1$.

Note that acyclicity is not a transitivity relation in the sense of (2.3.8), because infinitely many transitivity conditions are required to formulate this condition.

■

In the following example we investigate the transitivity-condition defined in (2.3.8) a little bit further in such a way that we will give some special transitivity conditions, which cannot lead to a set of relations that can be classified as a set of orderings.

<u>Example 2.3.10</u>

2.3.10.1 <u>Non-preserving of the irreversibility</u>

Consider the following transitivity condition:

$\langle \bar{a}\bar{s}, \bar{s}\rangle$-transitivity.

Suppose: R_A is $\langle \bar{a}\bar{s}, \bar{s}\rangle$-transitive,
$\langle x_0,x_1\rangle \in \bar{a}R_A$ and $\langle x_1,x_2\rangle \in \bar{s}R_A$.

Then it follows that $\langle x_0,x_2\rangle \in R_A$, which is very natural, but moreover it follows that $\langle x_2,x_0\rangle \in R_A$, since $\langle x_0,x_2\rangle \in \bar{s}R_A$. So $x_0 \longrightarrow x_1 \text{ — } x_2$ This is very odd.

Moreover $\langle x_1,x_1\rangle \notin R_A$, since this would lead to $\langle x_0,x_1\rangle \in \bar{s}R_A$ which contradicts our assumption.

It is easy to show that the set of irreflexive and $\langle \bar{a}\bar{s},\bar{s}\rangle$-transitive relations is not concatenationally closed. Hence, this set cannot be classified as a set of orderings. Notice that a path of type $\bar{a}\bar{s}$ is irreversible but a path of type $\bar{s}$ is reversible along R_A, i.e., the conversed path is along R_A.

2.3.10.2 <u>Non-conversibility</u>

Consider the following transitivity condition:

$\langle \bar{a}\bar{s}, \bar{a}\rangle$-transitivity.

Let $A = \{x,y,z\}$ $R_A = \{\langle x,y\rangle,\langle y,x\rangle,\langle y,z\rangle\}$.

Then R_A is $\langle \bar{a}\bar{s}, \bar{a}\rangle$-transitive, but $\bar{v}R_A$ is not $\langle \bar{a}\bar{s}, \bar{a}\rangle$-transitive.

The set of reflexive, $\langle \bar{a}\bar{s}, \bar{a}\rangle$-transitive, relations cannot be classified as a set of orderings, because it is not closed under conversion.

The set of $\langle \bar{a}\bar{s}, \bar{a}\rangle$-transitive and $\langle \bar{s}\bar{a}, \bar{a}\rangle$-transitive relations, however, can be classified as a set of orderings. This will be shown below.

2.3.10.3 <u>Discrimination between the diagonal and the reversible part of a relation</u>

Consider the following transitivity condition:

$\langle \bar{a}\bar{s}\bar{a}, \bar{a}\rangle$-transitivity.

Suppose R_A is $\langle \bar{a}\bar{s}\bar{a}, \bar{a}\rangle$-transitive.

If R_A is reflexive, then R_A is quasi-transitive.

If R_A is irreflexive, then R_A is not necessarily quasi-transitive.

Suppose R_A is $\langle \bar{a}\,\overline{ns}\,\bar{a}, \bar{a}\rangle$-transitive.

Then R_A need not be quasi-transitive.

Furthermore, the set of irreflexive and $\langle \bar{a}\bar{s}\bar{a}, \bar{a}\rangle$-transitive relations is not closed under substitution.

Let $R_A = \{\langle x,y\rangle, \langle y,z\rangle,\}_{\{x,y,z\}}$ and $R_B = \{\langle a,b\rangle, \langle b,a\rangle\}_{\{a,b\}}$,

where $|\{x,y,x,a,b\}| = 5$.

Then R_A : x.—>——.y—>——.z, R_B : a.———.b, and

$Sub(R_A, y, R_B)$: x.——>——.a——.b——>——.z

R_A and R_B are irreflexive and $\langle \bar{a}\bar{s}\bar{a}, \bar{a}\rangle$-transitive, but $Sub(R_A, y, R_B)$ is not $\langle \bar{a}\bar{s}\bar{a}, \bar{a}\rangle$-transitive. Notice that the $\langle \bar{a}\,\overline{ns}\,\bar{a}, \bar{a}\rangle$-transitivity discriminates between $\overline{nsca}R_A$ and $\bar{e}R_A$, (See example 2.2.4.2), for an arbitrary relation R_A, i.e., a pair $\langle x,y\rangle \varepsilon\ \bar{e}R_A$ will never occur as an actual step in a path of type $\bar{a}\,\overline{ns}\,\bar{a}$, whicle a pair $\langle a,b\rangle \varepsilon\ \overline{nsca}R_A$ may occur as such a step. This discrimination creates problems concerning the closedness under substitution.

Let $A = \{a,b,c\}$, $B = \{x,y\}$, $A \cap B = \varphi$,

$R_A = \bar{r}\langle\{\langle a,b\rangle,\langle b,c\rangle\},A\rangle$ and $R_B = \bar{r}\langle\{\langle x,y\rangle,\langle y,x\rangle\},B\rangle$. Then both R_A and R_B are $\langle\bar{a}\ \overline{ns}\ \bar{a}, \bar{a}\rangle$ - transitive, since there is no path of type $\bar{a}\ \overline{ns}\ \bar{a}$ along these relations. But $Sub(R_A, b, R_B)$ is not $\langle\bar{a}\ \overline{ns}\ \bar{a}, \bar{a}\rangle$ - transitive. This occurs since the preference type $\overline{ns}$ in the path $\bar{a}\ \overline{ns}\ \bar{a}$ cannot be transformed in a dummy way, i.e., there is no diagonal pair on such a path. Clearly R_A is not $\langle\bar{a}\bar{s}\bar{a}, \bar{a}\rangle$ - transitive, hence by this transitivity the closedness under substitution is not (yet) violated here.

■

In example 2.3.10 we discussed several types of transitivity conditions which by themselves do not lead to a set of relations that can be classified as a set of orderings. We will state some conditions for transitivity properties which exclude these odd transitivity properties discussed in (2.3.10). Furthermore, we will prove that transitivity conditions with these extra restrictions lead to a set of relations that can be classified as a set of orderings.

<u>Definition 2.3.11</u> <u>Restrictions on transitivity conditions.</u>

Let $w_1 = f_1f_2\ldots f_k$ and $w_2 = g_1g_2\ldots g_n$ be two words in $\hat{N}^+$, let $R_A \varepsilon$ Å and let $m \varepsilon M$.

2.3.11.1 $\langle w_1,w_2\rangle$-transitivity with respect to m is <u>irreversibility preserving</u> iff if there is a $t \varepsilon \{1,\ldots,k\}$, such that $\bar{a} \cap f_t \neq \bar{o}$, then there is a $j \varepsilon \{1,\ldots,n\}$, such that $\bar{a} \cap g_j \neq \bar{o}$.

2.3.11.2 $\langle w_1,w_2\rangle$-transitivity with respect to m is <u>dummy admittable</u> iff

for all $t \varepsilon \{1,2\ldots,k\}$, with $\overline{nsca} \cap f_t \neq \bar{o}$, $f_t \cup \bar{e} = f_t$, and for all $t \varepsilon \{1,2\ldots,n\}$, with $\bar{e} \cap g_t \neq \bar{o}$, $g_t \cup \overline{nsca} = g_t$.

2.3.11.3 R_A is $\langle w_1,w_2\rangle$-<u>classifiable transitive with respect to</u> m iff the following four hold:

2.3.11.3.1 R_A is $\langle w_1,w_2\rangle$-transitive with respect to m,

2.3.11.3.2 R_A is $\langle \bar{w_1}^{v},\bar{w_2}^{v}\rangle$-transitive with respect to m, 2.3.11.3.3 $\langle w_1,w_2\rangle$-transitivity with respect to m is irreversibility preserving, and

2.3.11.3.4 $\langle w_1,w_2\rangle$-transitivity with respect to m is dummy admittable.

■

A few remarks on definition 2.3.11:

+ $\langle w_1,w_2\rangle$-classifiable transitivity with respect to $\bar{\text{I}}$ is abbreviated by $\langle w_1,w_2\rangle$-classifiable transitivity.

+ Let w_1 and w_2 be two words in $\hat{N}^+$ such that $w_1 = f_1f_2...f_k$ and $w_2 = g_1g_2...g_n$.

Suppose the $\langle w_1,w_2\rangle$-transitivity with respect to $m' \in \tilde{M}$ is dummy admittable.

If the reversible part of $R_X \in \mathring{A}$, i.e., $\overline{nscaR_X}$, has something in common with f_tR_X, for a $t \in \{1,...,k\}$, then the diagonal is in f_tR_X, i.e., $\bar{e}R_X \subseteq f_tR_X$. If the diagonal has something in common with g_tR_X, for a $t \in \{1,...,n\}$, then the reversible part $\overline{nscaR_X}$ is in g_tR_X. It is straightforward to calculate that:

$\{f_1,f_2,...,f_k\} \subseteq \{\bar{o},\bar{d},\overline{dc},\bar{a},\bar{e},\bar{m},\overline{qm},\overline{rs},\overline{ra},\overline{rsc},\bar{r},\overline{sca},\overline{rcv},\overline{cav}\}$, and

$\{g_1,g_2,...,g_n\} \subseteq$

$\{\bar{o},\overline{ns},\bar{a},\overline{nsc},\bar{n},\overline{nsca},\overline{ncv},\overline{qscm},\overline{scm},\overline{ncav},\overline{sca},\overline{cqmv},\overline{cmv},\overline{cav}\}$.

+ Notice that (2.3.11.3.3) and (2.3.11.3.4) are conditions on w_1 and w_2 and not on R_A.

Now one of the important results of this section is proved.

<u>Theorem 2.3.12</u>

Let $w_1,w_2 \in \hat{N}^+$, let $m \in \tilde{M}$, and suppose

$V_1 := \{R_A$: R_A is reflexive & R_A is $\langle w_1,w_2\rangle$-classifiable transitive with respect to m$\} \neq \varphi$ and

$V_2 := \{R_A$: R_A is irreflexive & R_A is $\langle w_1,w_2\rangle$-classifiable transitive with respect to m$\} \neq \varphi$

Then V_1 and V_2 can be classified as sets of orderings.

Proof of theorem 2.3.12

By corollary 2.2.27 we have:

V_i is classifiable as a set of orderings iff

V_i^m is classifiable as a set of orderings, for $i \in \{1,2\}$.

$V_1^m := \{mR_A \in Å : mR_A$ is reflexive and mR_A is $\langle w_1,w_2\rangle$-classifiable transitive with respect to m$\}$.

Because of $m \in M$, it follows that $mmR_X = R_X$ for all $R_X \in Å$.

Hence:

$V_1^m := \{mR_A \in Å : mR_A$ is reflexive and R_A is $\langle w_1,w_2\rangle$-classifiable transitive$\}$

$= \{R_X \in Å : R_X$ is reflexive and R_X is $\langle w_1,w_2\rangle$-classifiable transitive$\}$.

Similarly it follows:

$V_2^m := \{R_X \in Å : R_X$ is irreflexive and R_X is $\langle w_1,w_2\rangle$-classifiable transitive$\}$.

Hence, it suffices to prove (2.3.12) for the case that $m = \bar{i}$.

We only prove that V_1 can be classified as a set of orderings. The proof for V_2 is similar.

By theorem 2.2.23 it is sufficient to prove:

$Id_{\{x\}} \in V_1$, for all $x \in U$, and V_1 closed under conversion, restriction, concatenation and substitution.

Proof of $Id_{\{x\}} \in V_1$ for all $x \in U$

Let $A \in Æ$ $|A| = 1$.

It suffices to prove Id_A is $\langle w_1,w_2\rangle$-transitive and Id_A is $\langle \bar{\check{w}}_1,\bar{\check{w}}_2\rangle$-transitive.

We only prove that Id_A is $\langle w_1,w_2\rangle$-transitive; the proof of the other transitivity is similar.

Suppose $\pi_1 = \langle x_0,x_1,x_2,\ldots,x_k\rangle$ is a path along $\tilde{Id}_A$ of type w_1. Then we have to prove that there is a $w_3 \in N^+$ embedded in w_2, such that there is a w_3-short cut π_2 of π_1.

Since $|A| = 1$, we have $x_0 = x_1 = x_2 = \ldots = x_k$.

Since $V_1 \neq \varphi$ it follows there is a $R_X \in V_1$.

By theorem 2.2.7, $\pi_3 = \langle a_0,a_1,a_2,\ldots,a_k\rangle$ is a path along R_X of type w_1, whenever $a_0 = a_1 = a_2 = \ldots = a_k$.

Hence, there is a π_4, a w_3-short cut of π_3, such that w_3 is embedded in w_2. $\pi_4 = \langle b_0,b_1,b_2,\ldots,b_t\rangle$.

Since $\{b_0,b_1,b_2,\dots,b_t\} \subseteq \{a_0,a_1,a_2,\dots,a_k\} = \{a_0\}$ it follows again by (2.2.7) that $\pi_5 = \langle x_0,\dots,x_t\rangle$ is such a w_3-short cut of π_1.

Proof of the closedness under conversion of V_1

Let $R_X \in V_1$.
It is sufficient to prove that $\bar{v}R_X$ is both $\langle w_1,w_2\rangle$-transitive and $\langle w_1^{\bar{v}},w_2^{\bar{v}}\rangle$-transitive.
Suppose: π_1 is a path of type w_1 along $\bar{v}R_X$.
Then $\pi_1^{\bar{v}}$ is a path of type $w_1^{\bar{v}}$ along R_X. (by lemma 2.3.7)
Hence, $\pi_1^{\bar{v}}$ can be cut short by a path π_2 along R_X of type w_3 embedded in $w_2^{\bar{v}}$.
Hence, by lemma 2.3.7, $\pi_2^{\bar{v}}$ is a $w_3^{\bar{v}}$-short cut along $\bar{v}R_X$ of π_1

such that $w_3^{\bar{v}}$ is embedded in w_2.
Hence, $\bar{v}R_X$ is $\langle w_1,w_2\rangle$-transitive.
Similarly it follows that $\bar{v}R$ is $\langle w_1^{\bar{v}},w_2^{\bar{v}}\rangle$-transitive.

Proof of the closedness under restriction of V_1

Let $R_X \in V$ and $Y \subseteq X$ $Y \neq \varphi$.
Suppose R_X is $\langle w_3,w_4\rangle$-transitive.
It is sufficient to prove that $R_{X|_Y}$ is $\langle w_3,w_4\rangle$-transitive.
Let π_3 be a path of type w_3 along $R_{X|_Y}$.
By the definition of $R_{X|_Y}$ and theorem 2.2.7 it follows:

π_3 is a path of type w_3 along R_X.

Since R_X is $\langle w_3,w_4\rangle$-transitive, there is a path π_5 along R_X such that π_5 is a w_5-short cut of π_3 and w_5 is embedded in w_4.
By the definition of a short cut, $R_{X|_Y}$ and (2.2.7) it follows that π_5 is a w_5-short cut of π_3 along $R_{X|_Y}$ and w_5 is embedded in w_4.

Proof of the closedness under concatenation of V_1

Let R_A, $R_B \in V_1$ with $A \cap B = \varphi$.
Suppose R_A and R_B are both $\langle w_3,w_4\rangle$-transitive, where this transitivity is irreversibility preserving.

Then it is sufficient to prove that R_A » R_B is $\langle w_3, w_4\rangle$-transitive.

Suppose π_3 is a path along R_A » R_B of type w_3;

$\pi_3 = \langle x_0, x_1, \ldots, x_k\rangle$.

Whenever x_0, x_k ε A or x_0, x_k ε B, π_3 is a path along R_A or R_B respectively, by theorem 2.2.7, the definition of a path and the definition of ».

Hence, suppose x_0 ε A and x_k ε B. (x_k ε A and x_0 ε B is impossible since the path π_3 is along R_A » R_B).

Then there is a x_i ε A and x_{i+1} ε B, with $f_{i+1} \cap \bar{a} \neq \varphi$, where $w_3 = f_1 \ldots f_k$.

Hence, there is a g_j, with $w_4 = g_1 \ldots g_j \ldots g_n$ and $g_j \cap \bar{a} \neq \varphi$.

Hence, $\langle x_0, x_k\rangle$ is a g_j-short cut of π_3 along R_A » R_B, such that g_j is embedded in w_4.

<u>Proof of the closedness under substitution of V_1</u>

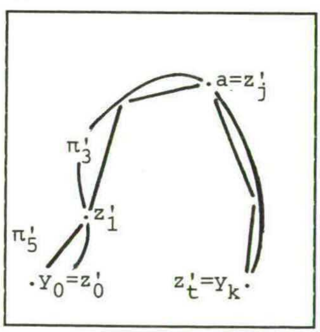
Suppose R^1_A, R^2_B ε V, with A ∩ B = φ and $\bar{v}R^2_B = R^2_B$, a ε A and $\langle w_3, w_4\rangle$-transitivity is dummy admittable.

Let $\pi_3 = \langle x_0, \ldots, x_k\rangle$ be a path of type $w_3 = f_1 \ldots f_k$ along R_Z, where $R_Z := \mathrm{Sub}(R^1_A, a, R^2_B)$.

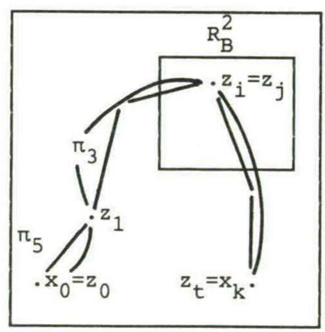
We have to prove:

There is a path π_5 along R_Z of type w_5 embedded in w_4, such that π_5 is a w_5-short cut of π_3.

R^1_A:



R_Z:



Take $\pi'_3 = \langle y_0, \ldots, y_k\rangle$, where $y_i = \begin{cases} x_i & \text{iff } x_i \in A - \{a\} \\ a & \text{iff } x_i \in B \end{cases}$

Proof of π_3' is a path along R_A^1 of type w_3

It is sufficient to prove that $\langle y_i, y_{i+1}\rangle \in f_{i+1}R_A^1$ for all $i \in \{0,\ldots,k-1\}$.

There are two cases.

Case 1 $\{y_t, y_{t+1}\} \cap A - \{a\} \neq \varphi$.

This case is simple to prove by (2.2.7) and the definition of R_Z.

Case 2 $\{y_t, y_{t+1}\} \subseteq \{a\}$.

Then $\{x_i, x_{i+1}\} \subseteq B$ and since R_B^2 is reversible it follows that $\langle x_i, x_{i+1}\rangle \in \overline{scaR_B^2}$. Hence, by the dummy admittablility of the $\langle w_3, w_4\rangle$-transitivity it follows that $\langle y_i, y_{i+1}\rangle = \langle a,a\rangle \in f_{i+1}R_A^1$.

Since π_3' is a path along R_A^1 of type w_3 and R_A^1 is $\langle w_3, w_4\rangle$-transitivity it follows that there is a path $\pi_5' = \langle z_0', z_1', \ldots, z_t'\rangle$ of type $w_5 = g_1g_2\ldots g_t$ embedded in w_4, such that π_5' is a w_5-short cut of π_3'.

Take $\pi_5 = \langle z_0, z_1, \ldots, z_t\rangle$, where $z_i = \begin{cases} z_i' & \text{iff } z_i' \in A - \{a\} \\ x_j & \text{iff } y_j = z_i' = a. \end{cases}$

Now it follows, similarly to the proof of the fact that π_3' is a path along R_A^1 of type w_3, that π_5 is a path along R_Z of type w_5.

This completes the proof.

■

As an immediate result we have:

Corollary 2.3.13

L(U) can be classified as a set of orderings.

Proof of corollary 2.3.13

$L(U) = V_1(U) \cap V_2(U) \cap V_3(U) \cap V_4(U)$.

$V_1(U)$, $V_2(U)$ and $V_4(U)$ are sets of relations which can be classified as sets of orderings (see lemma 3.1, 3.2 and 3.3)

$V_3(U) = \{R_A \in \mathring{A}: R_A$ is reflexive & R_A is $\langle \bar{a}^2, \bar{a}\rangle$-classifiable transitive$\}$.

Hence, by (2.2.28), (2.3.12), (2.3.1), (2.3.2) and (2.3.3) we are done.

■

We will show now that L(U) is contained in every set of relations which can be classified as a set of orderings. This result states that there is no set of relations which can be classified as a set of orderings and which is a non-trivial subset of L(U). Furthermore, since L(U) is contained in every set of relations, which can be classified, it is the most simple classifiable class of orderings. More precisely:

Theorem 2.3.14

Let $\varphi \neq V \neq Å$ be such that V can be classified as a set of orderings.

There is a $W \subseteq V$, such that $W \in \{L(U), L(U)^{\bar{q}}\}$.

Proof of theorem 2.3.14

It is sufficient to prove that $\{L(U), L(U)^{\bar{q}}\} \cap V \neq \varphi$.

There are two cases.

Case 1 R_X is reflexive for all $R_X \in V$.

By induction on $n \geq 1$ we prove:

for all $R_Y \in L(U)$, with $|Y| = n$, $R_Y \in V$.

Basis: $n = 1$

For all $x \in U$, $Id_{\{x\}} \in V$ because of the assumptions.

Induction step: Suppose $R_Y \in L(U)$ and $|y| = n+1$.

To prove that $R_Y \in V$ define

$Best(R_Y) := \{x \in Y: x \geq y: R_Y \text{ for all } y \in Y\}$.

By the transitivity, the completeness, the antisymmetry and the reflexivity of R_Y it follows that $Best(R_Y) = \{x\}$ for some $x \in Y$. Take $Y' = Y - \{x\}$. Then $R_Y = Id_X \gg R_{Y}|_{Y'}$ and we are done by the induction hypothesis, and by the closedness under restriction and concatenation of V.

Hence, $L(U) \subseteq V$.

Case 2 R_X is irreflexive for all $R_X \in V$.

Then $L(U) \subseteq V^{\bar{q}}$, hence $L(U)^{\bar{q}} \subseteq V$.

∎

Corollary 2.3.15

Let $\varphi \neq V \subseteq Å$ be such that V can be classified as a set of orderings. Then there is an unique $W \subseteq V$ such that $W \simeq L(U)$.

Proof of corollary 2.3.15

By theorem 2.3.14 and corollary 2.2.27 the existence of W is evident.

Suppose $W_1 \simeq L(U)$, $W_2 \simeq L(U)$, $W_1 \subseteq V$ and $W_2 \subseteq V$.

By (2.2.25) W_1 and W_2 can be classified as sets of orderings. By (2.3.14) there are $W_4, W_3 \in \{L(U), L(U)^{\bar{q}}\}$, such that $W_3 \subseteq W_1 \subseteq V$ and $W_4 \subseteq W_2 \subseteq V$.

Using (2.2.23) $W_3 = W_4$.

Obviously, $W_3 = W_1$ and $W_4 = W_2$.

Hence, $W_1 = W_2$, which completes the proof of the uniqueness.

■

§ 2.4 Minimal Extensions of L(U)

In § 2.3 we introduced L(U) as the set of reflexive, antisymmetric, complete and transitive relations. Furthermore, we proved that every set of relations which can be classified as a set of orderings, contains a set which is isomorphic to L(U). Hence, L(U) is the smallest non-empty set of relations which can be classified as a set of orderings. In this section several extensions of L(U) will be studied. Furthermore we will find all minimal extensions of L(U).

Let us start with some extensions of L(U) which might not be minimal.

$W(U) := V_2(U) \cap V_5(U)$,

$T(U) := V_2(U) \cap V_4(U)$, and

$Q(U) := V_2(U) \cap V_3(U)$,

where $V_5(U) := \{R_X \in \AA : R_X$ is reflexive and $\langle\overline{cav}\ \overline{cav},\overline{cav}\rangle$-classifiable transitive$\}$.

By the lemma's and theorems of the foregoing sections W(U), T(U) and Q(U) can be classified as sets of orderings.

Furthermore: W(U) is the set of all reflexive, complete and transitive relations, T(U) is the set of all reflexive, complete and antisymmetric relations and Q(U) is the set of all strongly complete and quasi-transitive relations. In literature (see Sen [1970], Roubens & Vincke [1985]) W(U) is called the set of <u>weak orderings</u>, T(U) is called the set of <u>tournaments</u> and Q(U) is called the set of <u>quasi-orderings</u>.

Evidently, $L(U) \subseteq Q(U)$, $L(U) \subseteq W(U)$ and $L(U) \subseteq T(U)$. Observe that for all $A \in Æ$ such that $|A| \geq 2$ it holds that $\bar{c}\varphi_A \in W(U)$ and $\bar{c}\varphi_A \in Q(U)$, but $\bar{c}\varphi_A \notin L(U)$. Hence, $L(U) \subset Q(U)$ and $L(U) \subset W(U)$. Furthermore, let $A = \{x,y,z\}$, $|A| = 3$ and $R_A := \bar{r}\langle\{\langle x,y\rangle,\langle y,z\rangle,\langle z,x\rangle\},A\rangle$, then $R_A \in T(U)$ and $R_A \notin L(U)$. Hence, $L(U) \subset T(U)$. So L(U) is a non-trivial subset of W(U), Q(U) and T(U).

We continue with formulating some definitions about sets of relations which can be classified as sets of orderings.

Definition 2.4.1

Let V and W be two sets of relations which can be classified as sets of orderings:

2.4.1.1 W is an extension of V iff there is a set W', such that $W' \subset W$ and $W' \cong V$, and

2.4.1.2 W is a minimal extension of V iff
W is an extension of V, and
if W' is an extension of V and $W' \subseteq W$ then $W' = W$.

■

If W is a minimal extension of V and $V \subset W$, then we write $V \underset{m}{\subseteq} W$.

Clearly Q(U),W(U) and T(U) are extensions of L(U). In the remaining part of this section we are going to find out which subsets in Q(U),W(U) and T(U) are minimal extensions of L(U) and, finally, we will investigate all minimal extensions of L(U). To do this, we explore our knowledge about minimal extensions. In fact, we will state a construction by which for any set of relations which can be classified as a set of orderings, every minimal extension of that set is determined. Furthermore, we characterize all these minimal extensions. First we introduce mechanisms which are closure operations on sets of relations.

Definition 2.4.2 Closure operations

Let $V \subseteq Å$ be a set of relations.

2.4.2.1 Σ_1 (V) := $\{R_A \varepsilon Å$: there is a $\sigma \varepsilon S_U$ and a $R_B \varepsilon V$, such that $\sigma R_B = R_A\}$ is the closure under permutation of V.

2.4.2.2 Σ_2 (V) := $\{R_A \varepsilon Å$: there is a $R_B \varepsilon V$, such that $\bar{v}R_B = R_A$ or $R_B = R_A\}$ is the closure under conversion of V.

2.4.2.3 Σ_3 (V) := $\{R_A \varepsilon Å$: there is a $R_B \varepsilon V$ and a $D \subseteq B$, with $\varphi \neq D$ and $R_{B|_D} = R_A\}$ is the closure under restriction of V.

2.4.2.4 Σ_4 (V) := $\{R_A \varepsilon Å$: there is a $k \varepsilon \{1,2,3...\}$ and there are $R_{A_1}, R_{A_2}, ... R_{A_k} \varepsilon V$, such that $R_A = R_{A_1} \gg R_{A_2} \gg ... \gg R_{A_k}\}$ is the closure under concatenation of V.

2.4.2.5 Σ_6 (U) := {R_A ε Å : there is a relation R_B ε V and there are relations $R_{A_1}, R_{A_2}, \ldots R_{A_k}$ ε Ω(V) such that:

(1) $A_1, A_2, \ldots A_k$ is a partition of A,

(2) there are distinct $b_1, b_2, \ldots b_k$ ε U such that
$B = \{b_1, b_2, \ldots b_k\}$, and

(3) for all <x,y> ε A x A:
<x,y> ε R_A iff there are i,j ε {1,2,3,...k}, such that either $i \neq j$, $\langle x,y\rangle \in A_i \times A_j$ and $\langle b_i,b_j\rangle \in R_B$, or i = j and <x,y> ε R_{A_i}

} is the <u>closure under substitution</u> of V, where Ω(V) is the set of <u>reversible relations</u> in V, i.e., $\Omega(V) = \{R_X \in V : \bar{v}R_X = R_X\}$.

■

The following theorem shows that the chosen names for Σ_1(V), Σ_2(V), Σ_3(V), Σ_4(V) and Σ_6(V) have the intended properties.

<u>Theorem 2.4.3</u>

Let V $\subseteq$ Å be a set of relations, then

2.4.3.1 Σ_1(V) is closed under permutation and V $\subseteq$ Σ_1(V),

2.4.3.2 Σ_2(V) is closed under conversion and V $\subseteq$ Σ_2(V),

2.4.3.3 Σ_3(V) is closed under restriction and V $\subseteq$ Σ_3(V),

2.4.3.4 Σ_4(V) is closed under concatenation and V $\subseteq$ Σ_4(V), and

2.4.3.5 Σ_6(V) is closed under substitution and V $\subseteq$ Σ_6(V).

Moreover, let V $\subseteq$ W $\subseteq$ Å be a set of relations, then

2.4.3.6 if W is closed under permutation, then Σ_1(V) $\subseteq$ W,

2.4.3.7 if W is closed under conversion, then Σ_2(V) $\subseteq$ W,

2.4.3.8 if W is closed under restriction, then Σ_3(V) $\subseteq$ W,

2.4.3.9 if W is closed under concatenation, then Σ_4(V) $\subseteq$ W, and

2.4.3.10 if W is closed under substitution, then Σ_6(V) $\subseteq$ W.

2.4.3.11 Furthermore $\Sigma_i\ \Sigma_j$ (V) $\subseteq$ $\Sigma_j\ \Sigma_i$ (V),
for all <i,j> ε {<1,1>,<1,2>,<1,3>,<1,4>,<1,6>,
<2,1>,<2,2>,<2,3>,<2,4>,<2,6>,
<3,1>,<3,2>,<3,3>,<3,4>,<3,6>,
<4,2>, <4,4>,
<6,2>, <6,4>,<6,6>}.

Proof of theorem 2.4.3

The proof of (2.4.3.1) up to (2.4.3.10) is elementary, but cumbersome and therefore left to the reader.

The proof of (2.4.3.11) is indicated by the following table, where in the cell <i,j> a reference to an assertion can be found by which the proof for the case <i,j> follows evidently. The assertions are trivially true.

	1	2	3	4	6
1	I	II	III	IV	V
2	II	I	VI	VII	VIII
3	III	VI	I	IX	X
4		VII		I	
6		VIII		IX	I

I Trivial, since $i = j$

II For all $R_A \in \mathring{A}$ and all $\sigma \in S_U$: $\sigma\bar{v}R_A = \bar{v}\sigma R_A$.

III For all $R_A \in \mathring{A}$ and all $B \subseteq A$, with $b \neq \varphi$, and all $\sigma \in S_U$: $\sigma(R_{A|_B}) = (\sigma R_A)|_{\sigma(B)}$.

IV For all $R_A, R_B \in \mathring{A}$, with $A \cap B = \varphi$, and all $\sigma \in S_U$: $\sigma(R_A \gg R_B) = \sigma(R_A) \gg \sigma(R_B)$.

V For all $R_A, R_B \in \mathring{A}$, with $A \cap B = \varphi$ and $\bar{v}R_B = R_B$, all $\sigma \in S_U$ and all $a \in A$: $\sigma\mathrm{Sub}(R_A, a, R_B) = \mathrm{Sub}(\sigma R_A, \sigma(a), \sigma R_B)$.

VI For all $R_A \in \mathring{A}$ and all $B \subseteq A$, with $B \neq \varphi$: $\bar{v}(R_{A|_B}) = (\bar{v}R_A)|_B$.

VII For all $R_A, R_B \in \mathring{A}$, with $A \cap B = \varphi$: $\bar{v}R_A \gg \bar{v}R_B = \bar{v}(R_A \gg R_B)$.

VIII For all $R_A, R_B \in \mathring{A}$, with $A \cap B = \varphi$ and $\bar{v}R_B = R_B$, and all $a \in A$: $\bar{v}\mathrm{Sub}(R_A, a, R_B) = \mathrm{Sub}(\bar{v}R_A, a, R_B)$.

IX For all $R_A, R_B \in \mathring{A}$, with $A \cap B = \varphi$, and all $C \subseteq A \cup B$, with

$$C \neq \varphi : (R_A \gg R_B)|_C = \begin{cases} R_{A|_C}, & \text{iff } C \subseteq A \\ R_{B|_C}, & \text{iff } C \subseteq B \\ R_{A|_{(A \cap C)}} \gg R_{B|_{(B \cap C)}}, & \\ \quad \text{iff } C \not\subseteq A \text{ and } C \not\subseteq B. & \end{cases}$$

X For all $R_A, R_B \in \mathring{A}$, with $A \cap B = \varphi$ and $\bar{v}R_B = R_B$, all $a \in A$ and all $C \subseteq A \cup B$, with $C \neq \varphi$:

$$\mathrm{Sub}(R_A,a,R_B)|_C = \begin{cases} R_A|_{(C-\{a\})}, \text{ iff } C \subseteq A \\ R_B|_C, \text{ iff } C \subseteq B \\ \mathrm{Sub}(R_A|_{(C \cup \{a\}) \cap A}, a, R_A|_{(B \cap C)}), \\ \quad \text{iff } C \not\subseteq A \text{ and } C \not\subseteq B \end{cases} .$$

XI For all $R_A, R_B, R_C \in \mathring{A}$, such that A,B and C are pairwise disjoint from each other and $\bar{v}R_C = R_C$, and all $a \in A \cap B$:

$$\mathrm{Sub}(R_A \gg R_B, a, R_C) = \begin{cases} \mathrm{Sub}(R_A,a,R_C) \gg R_B, \text{ iff } a \in A \\ R_A \gg \mathrm{Sub}(R_B,a,R_C), \text{ iff } a \in B. \end{cases}$$

■

Now we have the following corollary.

Corollary 2.4.4

Let $\varphi \neq V \subseteq \mathring{A}$ be a set of reflexive relations or a set of irrefexive relations.

Then $\Phi(V) := \Sigma_4(\Sigma_6(\Sigma_3(\Sigma_2(\Sigma_1(V)))))$ can be classified as a set of orderings.

Moreover, for all $V \subseteq W \subseteq \mathring{A}$, such that W can be classified as a set of orderings, it holds that $\Phi(V) \subseteq W$.

Proof of corollary 2.4.4

By theorem 2.4.3 it follows that:

$\Phi(V) \subseteq \Sigma_i(\Phi(V)) \subseteq \Phi(V)$ for all $i \in \{1,2,3,4,6\}$.

Hence, again by theorem 2.4.3, $\Phi(V)$ is closed under permutation, conversion, concatenation, restriction and substitution. The non-triviality of $\Phi(V)$ follows immediately from the (ir)reflexivity of all relations in $\Phi(V)$ and the closedness of $\Phi(V)$ under restriction, concatenation, and permutation.

Hence, $\Phi(V)$ can be classified as a set of orderings.

Suppose $V \subseteq W$ and W can be classified as a set of orderings.

Then by (2.4.3.6) up to (2.4.3.10) it follows that

$\Sigma_1(V) \subseteq W$, $\Sigma_2(\Sigma_1(V)) \subseteq W$, $\Sigma_3(\Sigma_2(\Sigma_1(V))) \subseteq W$,
$\Sigma_6(\Sigma_3(\Sigma_2(\Sigma_1(V)))) \subseteq W$ and $\Phi(V) \subseteq W$.
This completes the proof.

∎

So, if V is a set of reflexive or a set of irreflexive relations, then $\Phi(V)$ is the smallest set which can be classified as a set of orderings and which contains V. To construct a smallest set which can be classified as a set of orderings, and which contains a set V of (ir)reflexive relations, it is sufficient to close V with respect to the closure operations $\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4$ and Σ_6. Now, we will show that a minimal extension of a set X which can be classified as a set of orderings, can be obtained by closing a set X' with respect to concatenation and substitution, where $X \subseteq X'$ and $|X' - X| \leq 2$.

<u>Theorem 2.4.5</u>

Let $V \subseteq Å$ be classified as a set of orderings.
W is a minimal extension of V and $V \subset W$, iff
there is a $R_B \in W - V$, such that $|B| \geq 2$ and
for all $C \subset B$, with $C \neq \varphi$, $R_{B|_C} \in V$, and
$W = \Phi(V \cup \{R_B\}) = \Sigma_4\Sigma_6(V \cup \{\bar{v}R_B, R_B\})$.

<u>Proof of theorem 2.4.5</u>

Let $V \subseteq Å$ be classified as a set of orderings.
(if) Suppose $W = \Phi(V \cup \{R_B\}) = \Sigma_4\Sigma_6(V \cup \{\bar{v}R_B, R_B\})$, where
$R_B \in W - V$, such that for all $C \subset B$, with $C \neq \varphi$, $R_{B|_C} \in V$.
From corollary 2.2.4 it follows that W can be classified as set of orderings.
$R_B \in W - V$, so W is an extension of V.
Let $V', W' \subseteq Å$, such that $V' \simeq V$, $V' \subset W'$ and $W' \subseteq W$.
Since $V' \subset W'$ there is a $R'_D \in W' - V'$ such that for all
$C \subset D$, with $C \neq \varphi$, $R'_{D|_C} \in V'$.
Now $V' \subset \Phi(V' \cup \{R'_D\}) \subseteq W' \subseteq W$.
From (2.2.27.4), $V \simeq V'$ and $V' \subset W$ it follows that

$V' \in \{V, \overline{V^{qcv}}\}$.

We have two cases.

Case 1 $V' = V$.

$R'_D \in \Phi(V \cup \{R_B\})$ and $R'_D \notin V$.

Since $R'_D|_C \in V$ for all $C \subset D$, with $C \neq \varphi$, it follows that $R'_D \in \Sigma_2\Sigma_1(V \cup \{R_B\})$.

But then $R'_D \in \Sigma_2\Sigma_1(\{R_B\})$.

Hence, $W = \Phi(V \cup \{R_B\}) = \Phi(V \cup \{R'_D\}) \subseteq W'$.

Case 2 $V \neq V'$.

Then $V' = V^{\overline{qcv}}$ and $V' \neq V$.

Note that, if there are relations $R_X, R_Y \in V$, such that R_X is not antisymmetric and R_X is not complete, then $V^{\overline{cqv}} = V$.

Now it follows that either all relations in V are complete and all relations in V' are antisymmetric or all relations in V are antisymmetric and all relations in V' are complete.

Since $V' \neq V$ it follows, that $Y_7 \subseteq W$ or $Y_8 \subseteq W$ (See 2.2.20).

Hence, $R_D \in (\Sigma_1(\{R_B\}))^{\overline{qcv}}$, $R_D \in V$ and $R_B \in V'$. Hence, obviously $W = W'$.

(only if) Suppose $V \subset W$ and W is a minimal extension of V.

The existence of R_B follows evidently by a simple induction reasoning, the existence of $R'_D \in W - V$ and the finiteness of D.

By (2.2.4) it follows that $\Phi(V \cup \{R_B\})$ is classified as a set of orderings and $\Phi(V \cup \{R_B\}) \subseteq W$.

Since W is a minimal extension of V there is a set $V' \subset W$ such that $V' \simeq V$ and for all $W' \subseteq W$, which can be classified as a set of orderings, such that $V'' \subset W'$ and $V'' \simeq V$ it follows that $W' = W$.

Take $W' = \Phi(V \cup \{R_B\})$, then it follows that $W = \Phi(V \cup \{R_B\})$.

To complete the proof it is sufficient to prove that $\Sigma_4\Sigma_6(V \cup \{\bar{v}R_B, R_B\})$ can be classified as set of orderings $V \cup \{\bar{v}R_B, R_B\}$ is closed under restriction and conversion.

Hence, $V \cup \{\bar{v}R_B, R_B\} = \Sigma_3\Sigma_2(V \cup \{\bar{v}R_B, R_B\})$.

By theorem 2.4.3.11 it follows that for all $i \in \{2,3,4,6\}$:

$\Sigma_i\Sigma_4\Sigma_6\Sigma_3\Sigma_2(V \cup \{\bar{v}R_B, R_B\}) \subseteq \Sigma_4\Sigma_6\Sigma_3\Sigma_2(V \cup \{\bar{v}R_B, R_B\})$.

Hence, again by (2.4.3) it follows that for all $i \in \{2,3,4,6\}$:

$\Sigma_i\Sigma_4\Sigma_6\Sigma_3\Sigma_2(V \cup \{\bar{v}R_B,R_B\}) = \Sigma_4\Sigma_6\Sigma_3\Sigma_2(V \cup \{\bar{v}R_B,R_B\})$.

Hence, $\Sigma_4\Sigma_6(V \cup \{\bar{v}R_B,R_B\})$ is closed under conversion, restiction, substitution and concatenation.

Since either all relations in V are reflexive or all relations in V are irreflexive it follows by the definition of the Σ-closure operations that either all relations in $\Sigma_4\Sigma_6(V \cup \{\bar{v}R_B,R_B\})$ are reflexive or all relations in $\Sigma_4\Sigma_6(V \cup \{\bar{v}R_B,R_B\})$ are irreflexive.

Since V is classified it follows from (2.2.23) that it has property (2.2.23.1). From the definition of the Σ-closure operations it follows that $\Sigma_4\Sigma_6(V \cup \{\bar{v}R_B,R_B\})$ has property (2.2.23.1). Now by (2.2.23) we are done.

■

By theorem 2.4.5 it follows that a minimal extension W of a classifiable set V of orderings can be described by the closure, under several operations, of V united with a special chosen relation of W - V.

Supposing that a minimal extension corresponds with a minimal weakening of the conditions on the relations in V it follows that this weakening is brought about by one specific relation.

We will now prove that W(U) is a minimal extension of L(U).

Theorem 2.4.6

W(U) is a minimal extension of L(U).

Proof of theorem 2.4.6

Let $Y \in Æ$ such that $|Y| = 2$. It is sufficient to prove that $W(U) \subseteq \Sigma_4\Sigma_6(L(U) \cup \{\bar{c}\varphi_Y\})$.

Suppose $R_X \in W(U)$.

We have to prove that $R_X \in \Sigma_4\Sigma_6(L(U) \cup \{\bar{c}\varphi_Y\})$.

It is a well-known fact (see e.g. Roubens & Vincke) that by the completeness, the reflexivity and the transitivity of R_X there exist $X_1,X_2,....X_k$ in Æ such that $X_1,X_2,....X_k$ is a partition of X and

$R_X = \bar{c}\varphi_{X_1} \gg \bar{c}\varphi_{X_2} \gg ... \gg \bar{c}\varphi_{X_k} \in \Sigma_4\Sigma_6(L(U) \cup \{\bar{c}\varphi_Y\})$.

■

Corollary 2.4.7

Let $V \subseteq$ Å be a classifiable set of orderings, such that there is a $R_X \in V$, which is not complete or which is not antisymmetric.

Then there exists a subset W of V such that $W(U) \simeq W$.

Proof of corollary 2.4.7

Suppose V and R_X are as above.

Then there is a $m \in \tilde{M}$ such that $\bar{c}\varphi_Y \in V^m$ for some $Y \in$ Æ, with $|Y| = 2$.

Hence, $W(U) \simeq W(U)^m \subseteq V$.

■

We will now concentrate on another type of extension of L(U). First consider

$T_3(U) := \{R_X \in$ Å: R_X is reflexive &
R_X is antisymmetric &
R_X is complete &
R_X is $\langle\bar{a}^3, \overline{cav}\rangle$-classifiable transitive$\}$.

Evidently, $T_3(U)$ can be classified as a set of orderings. Let $R_X \in T_3(U)$, then there is no cycle of type $\bar{a}^4$ along R_X, since R_X is $\langle\bar{a}^3, \overline{cav}\rangle$-transitive. Let $X = \{x,y,z\}$ and $R_X = \bar{r}\langle\{\langle x,y\rangle,\langle y,z\rangle,\langle z,x\rangle\},X\rangle$, then $R_X \in T_3(U)$.

x.——>——.y——>——.z Hence, there are relations R_X in $T_3(U)$ such that there is a cycle of type $\bar{a}^3$ along R_X. Hence, $T_3(U)$ is an extension of L(U).

So $T_3(U)$ is the set of all tournaments in which circuits of length ≤ 3 are admitted.

Next we show that $T_3(U)$ is a minimal extension of L(U). This is established by several lemma's which by themselves give some further insight in the type of relations of $T_3(U)$. We start with a lemma which states that two cycles of type $\bar{a}^3$ are disjoint.

Lemma 2.4.8

Let $R_X \in T_3(U)$ and $\pi = \langle x_0,x_1,x_2,x_3\rangle$ be a cycle along R_X of type $\bar{a}^3$. Then the following holds:

2.4.8.1 for all $y \in X - \{x_0,x_1,x_2\}$ either for all $x \in \{x_0,x_1,x_2\}$ $\langle y,x\rangle \in \bar{a}R_X$ or for all $x \in \{x_0,x_1,x_2\}$ $\langle x,y\rangle \in \bar{a}R_X$, and

2.4.8.2 if $\pi' = \langle y_0,y_1,y_2,y_3\rangle$ is a cycle along R_X of type $\bar{a}^3$ and $\{y_0,y_1,y_2\} \neq \{x_0,x_1,x_2\}$, then $\{x_0,x_1,x_2\} \cap \{y_0,y_1,y_2\} = \varphi$.

Proof of lemma 2.4.8

Let R_X and π be as supposed above.

(2.4.8.1) Suppose $y \in X - \{x_0,x_1,x_2\}$.

Case 1 $\langle y,x_0\rangle \in \bar{a}R_X$

For reasons of symmetry it suffices to prove that $\langle y,x_2\rangle \in \bar{a}R_X$. Suppose $\langle x_2,y\rangle \in R_X$.
Then $\langle x_2,y\rangle \in \bar{a}R_X$ and $\langle x_2,y,x_0,x_1\rangle$ is a path of type $\bar{a}^3$.
Hence, $\langle x_2,x_1\rangle \in \bar{a}R_X$, since R_X is $\langle \bar{a}^3,\bar{a}\rangle$-transitive. But $\langle x_1,x_2\rangle \in \bar{a}R_X$ since π is a path of $\bar{a}^3$. The last two conclusions are contradicting each other.
Therefore $\langle y,x_2\rangle \in \bar{a}R_X$.

Case 2 $\langle x_0,y\rangle \in \bar{a}R_X$

Similar to case 1.

(2.4.8.2) Let $\langle x_0,x_1,x_2,x_0\rangle$ and $\langle y_0,y_1,y_2,y_0\rangle$ be as supposed above. Let $\{x_0,x_1,x_2\} \cap \{y_0,y_1,y_2\} \neq \varphi$.
Without loss of generality suppose:
$x_0 \notin \{y_0,y_1,y_2\}$, $y_0 \notin \{x_0,x_1,x_2\}$ and $x_1 = y_1$.
By (2.4.8.1) it follows that $\langle x_0,y_0\rangle \in \bar{a}R_X$ and $\langle y_0,x_0\rangle \in \bar{a}R_X$, since $\langle x_0,y_1\rangle \in \bar{a}R_X$ and $\langle y_0,x_1\rangle \in \bar{a}R_X$ resp..
Hence, we have a contradiction and are done.

■

The following lemma characterizes the relations of $T_3(U)$ in an other way. First we need a notation:
Let $X,Y \subseteq A$ be two disjoint non-empty sets and $R_A \in \mathring{A}$.
Then $X > Y : R_A$ iff $(X \times Y)_A \subseteq \bar{a}R_A$.

Lemma 2.4.9

Let $R_X \in Å$ such that R_X is reflexive, complete and antisymmetric. Then (2.4.9.1) and (2.4.9.2) are equivalent:

2.4.9.1 $R_X \in T_3(U)$.

2.4.9.2 There is a partition $C_1, C_2, \ldots C_k$ of X such that:

(a) for all $i,j \in \{1,2,\ldots k\}$, with $i < j$, $C_i > C_j: R_X$, and

(b) for all C_i, with $|C_i| \neq 1$, there are $x,y,z \in A$, such that $C_i = \{x,y,z\}$ and $\langle x,y,z,x \rangle$ is a path of type $\bar{a}^3$ along R_X.

Proof of lemma 2.4.9

(2.4.9.2) -> (2.4.9.1)

Evident, because if R_X satisfies condition (2.4.9.2), then R_X is $\langle \bar{a}^3, \bar{a} \rangle$-classifiable transitive.

(2.4.9.1) -> (2.4.9.2) Suppose $R_X \in T_3(U)$.

Let $W := \{C \subseteq A$: $|C| = 3$ and there are x,y,z such that $\{x,y,z\} = C$ and $\langle x,y,z,x \rangle$ is a path of type $\bar{a}^3$ along R_X, or there is a x such that $\{x\} = C$ and for all paths $\pi = \langle x_0,x_1,x_2,x_0 \rangle$ of type $\bar{a}^3$ along R_X $x \notin \{x_0,x_1,x_2\}\}$.

Then by (2.4.9.2) it follows that the elements of W form a partition of X. Let $W = \{D_1,D_2,\ldots,D_k\}$. By (2.4.9.1) it follows that for all $i,j \in \{1,\ldots k\}$, $i \neq j$ it holds that $D_i > D_j : R_X$ or $D_j > D_i : R_X$. Hence, it is sufficient to prove that for all $i_1,i_2,i_3 \in \{1,\ldots,k\}$ it holds that:

$[D_{i_1} > D_{i_2} : R_X \ \& \ D_{i_2} > D_{i_3} : R_X] \rightarrow [D_{i_1} > D_{i_3} : R_X]$.

Suppose $D_{i_1} > D_{i_2} :R_X$, $D_{i_2} > D_{i_3} :R_X$ and not $D_{i_1} > D_{i_3} :R_X$.

Then we have:

$D_{i_1} > D_{i_2} : R_X$, $D_{i_2} > D_{i_3} : R_X$ and $D_{i_3} > D_{i_1} : R_X$.

Since $D_{i_1} \neq \varphi$, $D_{i_2} \neq \varphi$ and $D_{i_3} \neq \varphi$ it follows that for all $x \in D_{i_1}$ there are $y \in D_{i_2}$ and $z \in D_{i_3}$ such that $\langle x,y,z,x \rangle$ is a path of $\bar{a}^3$ along R_X. Hence, by (2.4.9.2) it follows that $|D_{i_1}| = 3$ and $D_{i_1} = D_{i_2} = D_{i_3}$ which contradicts our assumptions.

■

By lemma 2.4.9 we see that a relation in $T_3(U)$ has cycles of length at most 3, which can be ordered linearly. Hence, the intransitivity of a relation in $T_3(U)$ is 'minimal' since the elements which build an intransitive cycle are not separated by elements not on that cycle (i.e., x,y is separated in R_X if $x > z > y : R_X$ for some $z \varepsilon X$) and the cycles have the smallest possible length 3.

Furthermore, let $X = \{x,y,z\} \varepsilon Æ$, $|X| = 3$ and $R_X = \bar{r}\langle\{\langle x,y\rangle,\langle y,z\rangle,\langle z,x\rangle\},X\rangle$, then $T_3(U) = \Sigma_4\Sigma_6(L(U) \cup \{R_X\})$. The following theorem states that $T_3(U)$ is a minimal extension of L(U).

<u>Theorem 2.4.10</u>

Let $W \subseteq Å$ be such that it can be classified as a set of orderings. Furthermore, let $R_A \varepsilon W$ such that R_A is reflexive, complete and antisymmetric. Then the following holds: if R_A is not $\langle\bar{a}^2,\bar{a}\rangle$-transitive, then $T_3(U) \subseteq W$.

<u>Proof of theorem 2.4.10</u>

From R_A as given above, we can construct $X = \{x,y,z\}$ and $R_X = \bar{r}\langle\{\langle x,y\rangle,\langle y,z\rangle,\langle z,x\rangle\},X\rangle$ such that $R_X \varepsilon W$.
Then evidently $T_3(U) = \Sigma_4\Sigma_6(L(U) \cup \{R_X\}) \subseteq W$.

■

<u>Corollary 2.4.11</u>

$T_3(U)$ is a minimal extension of L(U).

<u>Proof of Corollary 2.4.11</u>

Evident by theorem 2.4.10 and 2.4.5.

■

Another result evident from lemma 2.4.9 is that not $W(U) \simeq T_3(U)$, since for all $A \varepsilon Æ$, such that $|A| \geq 2$, it holds that $|\{R_A \varepsilon Å : R_A \varepsilon W(U)\}| \geq |\{R_A \varepsilon Å : R_A \varepsilon T_3(U)\}|$
Hence, we have two minimal extensions of L(U). The last theorem of this section states that these are the only minimal extensions of L(U).

Theorem 2.4.10

Let W be a minimal extension of $L(U)$, which can be classified as a set of orderings.

Then $W \simeq W(U)$ or $W \simeq T_3(U)$.

Proof of theorem 2.4.10

Without loss of generality (because of (2.2.23) and (2.2.27)) suppose R_X is reflexive for all $R_X \in W$.

By (2.3.14) it follows that $L(U) \subseteq W$.

Since W is an extension we have $L(U) \subset W$.

Now there is a $R_X \in W$, such that R_X is not antisymmetric or R_X is not complete or R_X is not transitive.

Case 1 R_X is not antisymmetric or R_X is not complete.

By (2.4.7) we have $W(U) \simeq W'$ for some $W' \subseteq W$.

Since W is a minimal extension of $L(U)$ ($\subset W(U)$) it must be such that $W' = W$.

Case 2 R_X is not transitive, R_X is antisymmetric and R_X is complete.

By (2.4.10) we have $T_3(U) \subseteq W$. Since $L(U) \subset T_3(U) \subseteq W$ and W is a minimal extension of $L(U)$ it follows that $W = T_3(U)$.

This completes the proof.

■

§ 2.5 Extensions of $T_3(U)$

Consider the inclusion diagram of minimal extensions at the end of § 2.6. In the previous section we have determined the most left inclusions between L(U) and $T_3(U)$ and between L(U) and W(U). In this section we will study extensions of $T_3(U)$. Hence, we will develop the top line of the diagram. In fact, we will only consider extensions of $T_3(U)$ which are subsets of T(U). Hence, we will investigate tournaments.

Since $T_3(U) \subset T(U)$ there exists extensions of $T_3(U)$, which are subsets of T(U).

First we will develop some notions introduced in graph theory (See Moon [1968]).

Definition 2.5.1

Suppose R_X, $R'_X \in Å$ and $\pi = \langle x_0, x_1, \ldots, x_k \rangle$ is a path along R_X in X from x_0 to x_k.

2.5.1.1 The mass of π is equal to $|\{x_0, x_1, \ldots, x_k\}|$.

Notation: $\text{mass}(\pi) := |\{x_0, x_1, \ldots, x_k\}|$.

2.5.1.2 π is a circuit iff $x_0 = x_k$ and $\text{mass}(\pi) = \text{length}(\pi)$.

2.5.1.3 π is a Hamilton-circuit (or spanning-circuit) along R_X iff π is a circuit of length $|X|$.

2.5.1.4 R_X is reducible iff there exists R_A, $R_B \in Å$, such that $A \neq \varphi$, $B \neq \varphi$, $A \cap B = \varphi$ and $R_A \gg R_B = R_X$.

2.5.1.5 R_X is irreducible iff R_X is not reducible.

2.5.1.6 $\delta(R_X, R'_X) := \frac{1}{2} |R_X \triangle R'_X| := \frac{1}{2} |(R_X - R'_X) \cup (R'_X - R_X)|$.

2.5.1.7 R_X is strongly connected iff for all $x,y \in X$ there is a path π along R_X from x to y.

∎

The mass of a path π is equal to the number of elements along which π leads. Hence, $\text{mass}(\pi) \leq \text{length}(\pi)$. π is a circuit, iff π is a cycle which visits only the starting point, which is also the end of π, twice and all the intermediate points precisely once. Hence, a Hamilton circuit visits all the elements once except the starting-point which is also visited at the end.

Now we have the following well-known characterization, developed by Rado [1943], Roy [1958], Camion [1959] and Harary & Moser [1966].

Theorem 2.5.2

(See also Moon [1968], theorem 2 pg. 5 and theorem 3 pg. 6). Let $R_X \varepsilon T(U)$. Then the following four statements are equivalent:

I for all $x \varepsilon X$ and for all $k \varepsilon \{3,4,\ldots,|X|\}$ there exists a circuit from x to x along R_X of length k,

II there exists a Hamilton-circuit along R_X,

III R_X is strongly connected, and

IV R_X is irreducible.

Proof of theorem 2.5.2

(I) -> (II) Trivial

(II) -> (III) Trivial

(III) -> (IV) Suppose R_X is reducible and $R_X = R_A \gg R_B$. Then for all $a \varepsilon A$ and $b \varepsilon B$ there is no path from b to a along R_X, since $A > B : R_X$. Hence, R_X is not strongly connected.

(IV) -> (I) The proof of this step is left to the reader. It can be found in Moon [1968], pg. 6 theorem 3.

■

Corollary 2.5.3

Let $R_X \varepsilon T(U)$. Then there is a unique partition $C_1,C_2,\ldots C_k$ of X, such that

$R_X = (R_X|_{C_1}) \gg (R_X|_{C_2}) \gg \ldots \gg (R_X|_{C_k})$ and

$R_X|_{C_i}$ is irreducible for all $i \varepsilon \{1,2,\ldots,k\}$

Proof of corollary 2.5.3

Let $R_X \varepsilon T(U)$.

Since X is finite we can find $R_{C_1}, R_{C_2},\ldots,R_{C_k}$ for some $k \geq 1$ such that: $R_X = R_{C_1} \gg R_{C_2} \gg \ldots \gg R_{C_k}$ and R_{C_i} is irreducible for every $i \varepsilon \{1,2,\ldots,k\}$.

Hence, $C_1,C_2,\ldots,C_k$ is a partition of X and $R_X|_{C_i} = R_{C_i}$ for all $i \varepsilon \{1,2,\ldots,k\}$.

Hence, $C_i > C_j : R_X$ for all $1 \le i < j \le k$.
$C_1, C_2, \ldots, C_k$ is a partition of X, for all $i \in \{1,2,\ldots,k\}$
$R_X|_{C_i}$ is irreducible and $R_X = (R_X|_{C_1}) \gg (R_X|_{C_2}) \gg .. \gg (R_X|_{C_k})$.
Suppose $D_1, D_2, \ldots, D_n$ is a partition of X, such that
$R_X = (R_X|_{D_1}) \gg (R_X|_{D_2}) \gg \ldots \gg (R_X|_{D_n})$ and $(R_X|_{D_i})$ is
irreducible for all $i \in \{1,2,\ldots n\}$.
Hence, for all $1 \le i < j \le k$ $C_i > C_j : R_X$ and
$1 \le t < s \le n$ $D_t > D_s : R_X$.
Hence, $C_1 \cap D_1 \neq \varphi$.
Suppose $C_1 \not\subseteq D_1$. Then $C_1 \cap D_1 > C_1 - D_1 : R_X|_{C_1}$.
Hence, $R_X|_{C_1}$ is reducible.
So $C_1 \subseteq D_1$. Similarly, $D_1 \subseteq C_1$.
Thus $C_1 = D_1$.
By induction it follows easily that $k = n$ and $D_i = C_i$ for all $1 \le i \le k = n$.
Hence, such a partition is unique.

■

Corollary 2.5.3 leads immediately to the following notion:

<u>Definition 2.5.4</u> <u>Irreducible partition</u>

Let $R_X \in T(U)$. $\langle R_{C_1}, R_{C_2}, \ldots, R_{C_k} \rangle$ is the <u>(finest) irreducible partitioning</u> of R_X iff $R_X = R_{C_1} \gg R_{C_2} \gg \ldots \gg R_{C_k}$ and R_{C_i} is irreducible for all $i \in \{1,2,\ldots k\}$.

■

The existence and uniqueness of such a finest irreducible partition of a tournament $R_X \in T(U)$ follows evidently from corollary 2.5.3.

We will define some sets of tournaments and prove that these sets can be classified as a set of orderings.

Let $k \ge 3$ and $l \ge 0$.

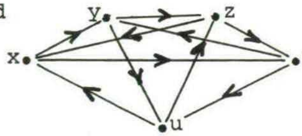
$T_{k,l}(U) := \{ R_X \in T(U)$: the finest irreducible partition $\langle R_{C_1}, R_{C_2}, \ldots, R_{C_t} \rangle$ of R_X satisfies for all $1 \le i \le t$:
(i) $|C_i| \le k$, and
(ii) there is a $R'_{C_i} \in L(U)$ with $\delta(R'_{C_i}, R_{C_i}) \le l$ $\}$.

Hence, $T_{k,l}(U)$ is the set of tournaments R_X, such that there is no circuit along R_X of length greater than k and the δ-distance between L(U) and every irreducible 'subrelation' $R_Y = R_{X|_Y}$, where $\varphi \neq Y \subseteq X$, is less than or equal to l.

Notice that $T_{k,0}(U) = L(U)$ for all $k \geq 3$.

Now $T_3(U) = T_{3,l}(U)$ for all $l \geq 1$ and if

a.——>——.b——>——.c : R_A and
└————<————┘


: R_X,

then R_A » $R_X \in T_{5,3}(U)$. Furthermore, note that if $R'_C \in L(U)$, $R_C \in T(U)$ is irreducible and $\delta(R'_C, R_C) = 1$, then $R_C = (R'_C - \langle\{\langle t,b\rangle\},C\rangle) \cup \langle\{\langle b,t\rangle\},C\rangle$, where $t = \text{best}(R'_C)$ and $b = \text{best}(\bar{v}R'_C)$ (See 2.3.14).

t.——>——.——>——... .b : R'_C and t.——>——.——>——... .b : R_C.
└————>————┘ └————<————┘

<u>Theorem 2.5.5</u>

Let $k \geq 3$, and $l \geq 0$.

$T_{k,l}(U)$ can be classified as a set of orderings.

<u>Proof of theorem 2.5.5</u>

Let $k \geq 3$ and $l \geq 0$.

Since $T_{k,l}(U) \subseteq T(U)$ it holds that: R_X is reflexive for all $R_X \in T_{k,l}(U)$.

Since $L(U) \subseteq T_{k,l}(U)$, (2.2.23.1) holds for $T_{k,l}(U)$.

Obviously $T_{k,l}(U)$ is closed under concatenation.

Since $\Omega(T_{k,l}(U)) = Y_1$ (See 2.2.20) and $T_{k,l}(U)$ is closed under permutation $T_{k,l}(U)$ is closed under substitution.

$T_{k,l}(U)$ is closed under conversion, since

(1) if $R_X \in T_{k,l}(U)$ and $\langle R_{C_1}, R_{C_2}, \ldots, R_{C_t}\rangle$ is an irreducible partition of R_X, then evidently $\langle \bar{v}R_{C_t}, \ldots, \bar{v}R_{C_2}, \bar{v}R_{C_1}\rangle$ is the finest irreducible partitioning of $\bar{v}R_X$, and

(2) for all $R^1_X, R^2_X \in Å$ it holds: $\delta(R^1_X, R^2_X) = \delta(\bar{v}R^1_X, \bar{v}R^2_X)$.

$T_{k,l}(U)$ is closed under restriction, since

$\delta(R_X^1, R_X^2) \geq \delta(R_X^1|_Y, R_X^2|_Y)$ for all R_X^1, R_X^2 ε Å and all Y ε Æ, such that $Y \subseteq X$.

Hence, we are done by theorem 2.2.23.

∎

In the following theorem several inclusions between classified sets of tournaments will be proved.

<u>Theorem 2.5.6</u>

Suppose $\{k,m,n\} \subseteq \{1,3,4,5,...\}$, $\{l,i,j\} \subseteq \{0,1,2,3,...\}$, $m > n$ and $i > j$.

2.5.6.1 If $l > 0$, then $T_{n,l}(U) \subset T_{m,l}(U)$.

2.5.6.2 If $l = 0$, then $L(U) = T_{m,l}(U) = T_{n,l}(U)$.

2.5.6.3 $T_{k,l}(U) \subset T(U)$.

2.5.6.4 If $k = 1$, then $T_{k,i}(U) = T_{k,j}(U) = L(U)$.

2.5.6.5 If $k = 3$ and $j \leq i$, then $T_{3,i}(U) = T_{3,j}(U) = T_3(U)$.

2.5.6.6 If $k = 4$ and $j \leq i$, then $T_{4,i}(U) = T_{4,j}(U) =: T_4(U)$.

2.5.6.7 If $j \geq \binom{k}{2}$, then $T_{k,i}(U) = T_{k,j}(U) =: T_k(U)$.

2.5.6.8 If $l \geq 1 + \sum_{5 \leq t \leq k} \text{round}(\frac{1}{2} \cdot (t-1))$ and $k \geq 3$, then $T_{k,l}(U) = T_k(U)$, where $\text{round}(x) = t$ iff $t \in \{0,1,...\}$ and $t \leq x < t + 1$.

<u>Proof of theorem 2.5.6</u>

(2.5.6.1) Let X ε Æ with $|X| = m$.

Take $R_X \in L(U)$. Suppose $x_1x_2...x_m : R_X$.

Take $R'_X = (R_X - \langle\{\langle x_1,x_m\rangle\},X\rangle) \cup \langle\{\langle x_m,x_1\rangle\},X\rangle$.

By definition of $T_{k,l}(U)$ we have $T_{n,l}(U) \subseteq T_{m,l}(U)$.

Since $\delta(R'_X,R_X) = 1$ and $l \geq 1$ it follows that:

$R'_X \in T_{m,l}(U)$ and $R'_X \notin T_{n,l}(U)$.

Hence, $T_{n,l}(U) \subset T_{m,l}(U)$.

(2.5.6.2) Trivial, since the set of irreducible tournaments in L(U) is equal to Y_1 (See 2.2.20).

(2.5.6.3) By definition $T_{k,l}(U) \subseteq T(U)$.

By (2.5.6.1) we have $T_{k,l}(U) \subset T_{k+1,l}(U) \subseteq T(U)$ if $l > 0$.

By (2.5.6.2) it follows $T_{k,l}(U) = L(U) \subset T(U)$ if $l = 0$.

(2.5.6.4) Trivial.

(2.5.6.5) By definition it follows
$L(U) \subseteq T_{3,j}(U) \subseteq T_{3,i}(U) \subseteq T_3(U)$. Hence, by corollary 2.4.11 we are done.

(2.5.6.6) Notice that for all irreducible $R_X \in T(U)$, such that $|X| \leq 4$ the following holds:
there exists $R'_X \in L(U)$ such that $\delta(R'_X, R_X) \leq 1$.
Hence, we are done by the definition of $T_{k,1}(U)$.

(2.5.6.7) Note that for all relations R_X, $R'_X \in T(U)$ the following holds: $\delta(R_X,R'_X) \leq \binom{|X|}{2}$.
Hence, by this and the definition of $T_{k,1}(U)$ we are done.

(2.5.6.8) It is sufficient to prove by induction on k that:
for all $R_X \in T_{k,1}(U)$, with $3 \leq k = |X|$ and $1 \geq 1$,
there is a $R'_X \in L(U)$, with $\delta(R_X,R'_X) \leq 1 + \sum_{5 \leq i \leq k} \text{round}(\frac{1}{2}.(i-1))$.

<u>Basis</u>: $k \in \{3,4\}$ is evident.

<u>Induction step</u>: Take $R_Y \in T_{k+1,1}(U)$, $|Y| = k+1$, $1 \geq 1$.
Take $x \in Y$, $X = Y - \{x\}$, $R_X := R_{Y|_X}$ and $R'_X \in L(U)$ such that $\delta(R_X,R'_X) \leq 1 + \sum_{5 \leq t \leq k} \text{round}(\frac{1}{2}.(t-1))$.
Without loss of generality suppose
$|\{a \in Y : a > x : R_Y\}| \geq |\{a \in Y : x > a : R_Y\}|$.
Hence, $\text{round}(\frac{1}{2} . k) \geq |\{a \in Y : x > a : R_Y\}|$.
Take $R'_Y = R'_X \gg \text{Id}_{\{x\}}$. Then $R'_Y \in L(U)$ and
$\delta(R_Y,R'_Y) = \delta(R_X,R'_X) + |\{a \in Y : x > a : R_Y\}| \leq$
$1 + \sum_{5 \leq t \leq k+1} \text{round}(\frac{1}{2}.(t-1))$.

■

In theorem 2.4.6 several classified sets of tournaments have been discussed. It is at first sight odd that the number 2 is excluded from the range of k, m and n, but since by the antisymmetry of tournaments cycles of length 2 are excluded it is evident that 2 is not in that range.

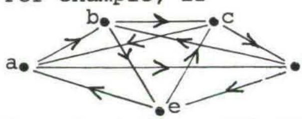
We introduce a third sequence of sets of tournaments. Let $k \in \{1,3,4,5,...\}$, $m,n \in \{0,1,2,...\}$ and $m > n$,
$T_{k,m,n}(U) := \{ R_X \in T(U) :$
the finest irreducible partition $\langle R_{C_1}, R_{C_2}, R_{C_3},..., R_{C_t}\rangle$ of R_X, satisfies the following three conditions:

(1) for all $1 \le i \le t$, $|C_i| \le k$,

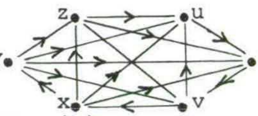
(2) for all $1 \le i \le t$, with $|C_i| < k$, there is a $R'_C \in L(U)$, such that $\delta(R_{C_i}, R'_{C_i}) \le m$, and

(3) for all $1 \le i \le t$, with $|C_i| = k$, there is a $R'_C \in L(U)$, such that $\delta(R_{C_i}, R'_{C_i}) \le n$ }.

For example, if



$: R_A$ and



$: R_X$,

then $R_A \gg R_X \in T_{6,3,2}(U)$ and $R_A \gg R_X \notin T_{6,2}(U)$.

Of course $T_{k,m,n}(U)$ suggests the existence of more odd sequences of sets of tournaments. The reason of introducing these sets is given by the following theorem. We will not investigate tournaments further, not because of their difficulty, oddness or not being of sufficient interest, but because we will only develop in this chapter a classification mechanism and show that this mechanism is meaningful.

<u>Theorem 2.5.7</u>

For all $k \in \{1,3,4,5,\ldots.\}$, $l,m,n \in \{0,1,2,\ldots\}$, such that $l > m > n$ it holds that:

2.5.7.1 $T_{k,m,n}(U)$ can be classified as a set of orderings, and

2.5.7.2 $T_{k,l,1}(U)$ is a minimal extension of $T_{k-1,l}(U)$.

<u>Proof of theorem 2.5.7</u>

(2.5.7.1) The proof of (2.5.7.1) is similar to the proof of theorem 2.5.5. Therefore it is left to the reader.

(2.5.7.2) Note that $l > 1$ and $T_{k-1,l}(U) \subset T_{k,l,1}(U)$.
Let $X \in$ Æ be such that $|X| = k$. Let $R_X \in L(U)$.
Take $t = \text{best}(R_X)$ and $b = \text{best}(\bar{v}R_X)$ (See 2.3.14).
t is the top element, it is most preferred, and
b is the bottom element, it is least preferred.
Take $R'_X = (R_X - \langle\{\langle t,b\rangle\},X\rangle) \cup \langle\{\langle b,t\rangle\},X\rangle$.
By (2.5.2) it follows that R'_X is irreducible, since there is a Hamilton-circuit along R_X.
Obviously R'_X is the only type of irreducible relations, such that $|X| = k$ and there is a relation $R''_X \in L(U)$, such that

$\delta(R'_X, R''_X) = 1$. (If we do not converse the top and bottom element of a linear ordering, it is reducible).

Hence, by the definition of $T_{k,1,\underline{1}}(U)$ it follows:

$T_{k,1,1}(U) = \Sigma_4\Sigma_1(T_{k-1,1}(U) \cup \{R'_X, \bar{v}R'_X\})$

$\qquad\quad = \Sigma_4\Sigma_6(T_{k-1,1}(U) \cup \{R'_X, \bar{v}R'_X\})$.

Hence, by (2.4.5) we are done.

■

Let us make some final remarks on the sets of tournaments which can be classified as sets of orderings:

+ First of all we observe the following sequence of sets of tournaments which can be classified as sets of orderings: $L(U) \subset T_3(U) \subset T_4(U) \subset T_5(U) \ldots. \subset T_k(U) \subset T_{k+1}(U) \ldots \subseteq T(U)$, where $T_k(U)$ is the set of all tournaments in which circuits of length $\leq k$ are admitted. (For a more precise definition of $T_k(U)$ see (2.5.6.5), (2.5.6.6) and (2.5.6.7)). This sequence is infinite. Furthermore $T_{k+1}(U)$ is an extension of $T_k(U)$ for all $k \geq 3$. Observe that admitting circuits of length less than or equal to k+1 in a relation is a weakening of the "transitivity-property": admitting circuits of length less than or equal to k, it is reasonable after all that $T_{k+1}(U)$ is an extension of $T_k(U)$.
+ Consider the sequence described above. By corollary 2.4.6 $T_3(U)$ is a minimal extension of $L(U)$. By theorem 2.5.7 $T_{k,\binom{k-1}{2},1}(U)$ is a minimal extension of $T_{k-1,\binom{k-1}{2}}(U) = T_{k-1}(U)$. By theorem 5.6.2 we have $T_{4,\binom{3}{2},1}(U) = T_4(U)$. Hence, $T_4(U)$ is a minimal extension of $T_3(U)$. Furthermore it is easy to prove that $T_{k,\binom{k-1}{2},1}(U) \subset T_k(U)$. Denoting a minimal extension in an inclusion by the symbol $\underset{m}{\subset}$ it follows that:

  $L(U) \underset{m}{\subset} T_3(U) \underset{m}{\subset} T_4(U) \underset{m}{\subset} T_{5,\binom{4}{2},1}(U) \subset T_5(U) \underset{m}{\subset} T_{6,\binom{5}{2},1}(U) \ldots.$
  $\ldots \subset T_k(U) \underset{m}{\subset} T_{k+1,\binom{k}{2},1}(U) \subset T_k(U) \subsetneq \ldots. T(U)$.

  Of course, there are a lot of other such sequences.
+ Notice δ is a distance function on Å. Hence, the requirement that there is a $R'_X \in L(U)$, such that $\delta(R_X, R'_X) \leq k$, can be

interpreted as to mean that the distance of R_X to L(U) is less than or equal to k.

Hence, comparing $T_{k,\binom{k}{2}}(U)$ with $T_{k+1,\binom{k}{2},1}(U)$, we observe that the "transitivity property" for relations in the former set is stronger than in the latter, since in the latter relations circuits of length k+1 are admitted. Actually, this weakening of the "transitivity property" is minimal, since circuits of length k+1 resemble linear orderings very much and in linear orderings the "transitivity" is "perfect" (See theorem 2.3.15). Hence, after all, it is reasonable that $T_{k,\binom{k}{2},1}(U)$ is a minimal extension of $T_{k+1,\binom{k}{2}}(U)$.

§ 2.6 Extensions of W(U)

In this section we will study extensions of W(U). We will find interesting sets of relations, which can be classified as sets of orderings, such as: the set of semi-orderings and the set of interval orderings. Furthermore, we will try to find minimal extensions of these sets. As in §2.5, these minimal extensions are based on a minimal weakening of the transitivity condition on relations. At the end of this section an inclusion diagram of classified sets of orderings has been drawn. In this section the bottom line of this diagram will be explored. So in this section we are mainly concerned with subsets of Q(U) (See the beginning of §2.4) the set of reflexive, complete and quasi-transitive relations. In literature Q(U), $Q(U)^{\overline{c}}$ (the set of asymmetric and quasi-transitive relations) and $Q(U)^{\overline{qc}}$ (the set of reflexive, antisymmetric and quasi-transitive relations) are called the set of quasi-orderings. Since $Q(U) \simeq Q(U)^{\overline{c}} \simeq Q(U)^{\overline{qc}}$ this fact, of naming equally different sets, is natural in order theoretic terms. We call Q(U) the set of quasi-orderings, because Q(U) is used as such in chapter 4.

We start with the set of interval orderings.

Let $I(U) := \{R_X \in \text{Å}: R_X$ is complete, reflexive and PIP-transitive$\}$ (See 2.2.8.14).

Hence, $I(U) := \{R_X \in \text{Å}: R_X$ is complete, reflexive and $\langle \overline{a}\ \overline{rs}\ \overline{a}, \overline{a}\rangle$-transitive$\}$.

I(U) is called the set of (total) <u>interval orderings</u> (See also Roubens & Vincke [1985], Blair & Pollack [1979] and Blau [1979]). I(U) can be classified as a set of orderings. This is easily proved by lemma 2.3.1 and 2.3.2 and theorem 2.3.12 and 2.2.28. Furthermore, $I(U)^{\overline{c}}$ is often called the set of <u>strict interval orderings</u>.

The reader who wants to have more background information on interval orderings and semi-orderings is referred to Roberts [1979], Fishburn [1973], Luce [1956] and many others which can be found in Roberts [1979].

It is evident that $L(U) \subset W(U) \subset I(U) \subset Q(U)$.
Hence, $I(U)$ is an extension of $W(U)$.

Another well-known set of orderings is the set of (*total*) *semi-orderings*, defined as follows (See Roubens & Vincke [1985], Blair & Pollack [1979] and Blau [1979]):

$S(U) := \{R_X \in$ Å: R_X is complete, reflexive,
$\langle \bar{a}\ \overline{rs}\ \bar{a},\ \bar{a}\rangle$-transitive and
$\langle \bar{a}\ \bar{a}\ \overline{rs},\ \bar{a}\rangle$-transitive$\}$.

Since $\langle \bar{a}\ \bar{a}\ \overline{rs},\ \bar{a}\rangle$-transitivity implies $\langle \overline{rs}\ \bar{a}\ \bar{a},\ \bar{a}\rangle$-transitivity it follows that:

$S(U) := \{R_X \in$ Å: R_X is complete, reflexive,
$\langle \bar{a}\ \overline{rs}\ \bar{a},\ \bar{a}\rangle$-classifiable transitive and
$\langle \bar{a}\ \bar{a}\ \overline{rs},\ \bar{a}\rangle$-classifiable transitive$\}$.

Hence, again applying lemma 2.3.1 and 2.3.2 and theorem 2.3.12 and 2.2.28 it follows that $S(U)$ is a set of relations which can be classified as a set of orderings. Furthermore, it follows evidently that $L(U) \subset W(U) \subset S(U) \subset I(U) \subset Q(U)$. $S(U)^{\bar{c}}$ is often called the set of *strict semi-orderings*. $S(U) \simeq S(U)^{\bar{c}}$.

Let us examine the weakening of the transitivity conditions going from $W(U)$ to $Q(U)$. Notice that:

for all $R_X \in W(U)$ it holds that:

R_X is $\langle \bar{a}\ \bar{a},\ \bar{a}\rangle$-transitive,
R_X is $\langle \overline{rs}\ \overline{rs},\ \overline{rs}\rangle$-transitive,
R_X is $\langle \bar{a}\ \bar{a}\ \overline{rs},\ \bar{a}\rangle$-transitive,
R_X is $\langle \bar{a}\ \overline{rs}\ \bar{a},\ \bar{a}\rangle$-transitive and
R_X is $\langle \overline{rs}\ \bar{a}\ \bar{a},\ \bar{a}\rangle$-transitive,

for all $R_X \in S(U)$ it holds that:

R_X is $\langle \bar{a}\ \bar{a},\ \bar{a}\rangle$-transitive,
R_X is $\langle \bar{a}\ \bar{a}\ \overline{rs},\ \bar{a}\rangle$-transitive,
R_X is $\langle \bar{a}\ \overline{rs}\ \bar{a},\ \bar{a}\rangle$-transitive and
R_X is $\langle \overline{rs}\ \bar{a}\ \bar{a},\ \bar{a}\rangle$-transitive,

for all $R_X \varepsilon I(U)$ it holds that:

R_X is $<\bar{a}\ \bar{a},\ \bar{a}>$-transitive and

R_X is $<\bar{a}\ \overline{rs}\ \bar{a},\ \bar{a}>$-transitive, and,

for all $R_X \varepsilon Q(U)$ it holds that:

R_X is $<\bar{a}\ \bar{a},\ \bar{a}>$-transitive.

Hence, by going from W(U) to S(U) we drop the $<\overline{rs}\ \overline{rs},\ \overline{rs}>$-transitivity, by going from S(U) to I(U) we drop the $<\bar{a}\ \bar{a}\ \overline{rs},\ \bar{a}>$-transitivity and by going from I(U) to Q(U) we drop the $<\bar{a}\ \overline{rs}\ \bar{a},\ \bar{a}>$-transitivity. We have shown above that S(U) is an extension of W(U), I(U) is an extension of S(U) and Q(U) is an extension of I(U). In the remaining part of this section we will show that neither of these extensions is minimal, so that there are other weakenings of the transitivity conditions which lead to smaller extensions. In fact we will give minimal extensions of these sets of relations and hope to bring more insight into the structure of these sets.

We start with a minimal extension of W(U). In our research of minimal extensions of W(U), S(U) and I(U) we restrict ourselves to minimal extensions in Q(U). Before mentioning several sets of relations which can be classified as sets of orderings, we will develop some basic notions about relations in I(U) and S(U). First we define the notion of maximal indifference class.

<u>Definition 2.6.1</u> <u>Maximal indifference class</u>

Let $R_X \varepsilon Å$ and $Y \subseteq X$.

Y is an <u>indifference class</u> of R_X iff $\bar{c}\varphi_Y \subseteq \bar{s}R_X$.

Y is a <u>maximal indifference class</u> of R_X iff Y is an indifference class of R_X and for all indifferent classes Y' of R_X, $Y \not\subset Y'$.

■

If Y is a maximal indifference class of R_X, then all the elements in Y are indifferent to each other with respect to R_X, and there is no Y' in X such that $Y \subset Y'$ and all the elements in

Y' are indifferent to each other. This explains the name maximal indifference class. Such a set is also called "maximal clique" (See Roubens & Vincke [1985]), a name which is related to graph-theory.

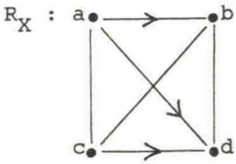
<u>Example 2.6.2</u>

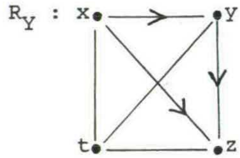
If $R_X \in L(U)$, then its maximal indifference classes are singletons which can be ordered linearly.

If $R_X \in W(U)$, then its maximal indifference classes are not necessarily singletons, but they are still disjoint and can be ordered linearly.

These two facts are well-known in literature (See the preceding sections). We will show that maximal indifference classes lose properties when going from W(U) via S(U) to I(U).

Take the following relations:





$R_X \in S(U)$ $\qquad\qquad$ $R_Y \in I(U) - S(U)$.

The maximal indifference classes of R_X are:

{a,c}, {c,b} and {b,d}.

Note that these classes are not disjoint, but can be ordered (this is shown in the next lemma). Furthermore, the intersection of three of such classes is empty.

The maximal indifference classes of R_Y are:

{x,t}, {y,t} and {z,t}.

Again the classes are not disjoint, and can be ordered.

But the intersection of three of such classes is not empty.

■

In the example above we showed that the formal weakening of the transitivity conditions for relations going from L(U) via W(U) and S(U) to I(U) has an intuitive foundation, which can be expressed by conditions on the maximal indifference classes. We will state a formal result on this subject:

Lemma 2.6.3

Let $R_X \in Å$.

2.6.3.1 $R_X \in I(U)$ iff (2.6.3.5) (a),(b) and (c) hold
iff (2.6.3.5) (a) and (b) hold.

2.6.3.2 $R_X \in S(U)$ iff (2.6.3.5) (a),(b),(c) and (d) hold.

2.6.3.3 $R_X \in W(U)$ iff (2.6.3.5) (a),(b),(c) and (e) hold.

2.6.3.4 $R_X \in L(U)$ iff (2.6.3.5) (a),(b),(c),(e) and (f) hold.

2.6.3.5 There exists a set of maximal indifference classes of R_X, say $\{C_1, C_2, \ldots C_t\}$ such that:

(a) $\cup \{C_i : 1 \leq i \leq t\} = X$,

(b) for all $1 \leq i < j \leq t$: $(C_i - C_j) > (C_j - C_i) : R_X$,

(c) for all $1 \leq i < j \leq t$: $C_i \cap C_j = \cap \{C_t : i \leq t \leq j\}$,

(d) for all $1 < i+1 < j \leq t$: $C_i \cap C_j = \varphi$,

(e) for all $1 \leq i < j \leq t$: $C_i \cap C_j = \varphi$, and

(f) for all $1 \leq i \leq t$: $|C_i| = 1$.

■

Although lemma 2.6.3 is partly well-known in literature especially 2.6.3.1, 2.6.3.3 and 2.6.3.4, we will prove 2.6.3.1 and 2.6.3.2 to make the reader familiar with these orderings.

Proof of lemma 2.6.3

(2.6.3.1) (if) The strong completeness of R_X follows evidently from (2.6.3.5) (a) and (b).

Suppose: $a > b : R_X$, $(bc) : R_X$ and $c > d : R_X$.

It is sufficient to prove that $a > d : R_X$.

By (2.6.3.5) (a) there are $C_{i_1}, C_{i_2}, C_{i_3} \in \{C_1, C_2, \ldots, C_k\}$ such that: $a \in C_{i_1}$, $b,c \in C_{i_2}$, $d \in C_{i_3}$, $b \notin C_{i_1}$ and $d \notin C_{i_2}$.

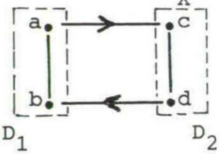
Clearly by (2.6.3.5 b) $i_1 < i_2 < i_3$.

Hence, $a > d : R_X$, since $a \in C_{i_1} - C_{i_3}$ and $d \in C_{i_3} - C_{i_1}$.

(only if) (2.6.3.5 a) is trivial.

(2.6.3.5 b) First we prove that for all $D_1, D_2 \in \{C_1, C_2, \ldots C_k\}$
$(D_1 - D_2) > (D_2 - D_1) : R_X$ or $(D_2 - D_1) > (D_1 - D_2) : R_X$ (2.6.3.6).

Suppose: $a, b \in D_1$ and $c, d \in D_2$, such that $a > c : R_X$ and $d > b : R_X$.



By the $\langle \bar{a}\ \overline{rs}\ \bar{a}, \bar{a}\rangle$-transitivity it follows that $a > b : R_X$.
This contradicts our assumption.
Hence, (2.6.3.6) holds.
Suppose $D_1, D_2, D_3 \in \{C_1, \ldots, C_k\}$ such that
$(D_1 - D_2) > (D_2 - D_1) : R_X$ and $(D_2 - D_3) > (D_3 - D_2) : R_X$.
It is sufficient to prove that $(D_1 - D_3) > (D_3 - D_1) : R_X$.
By the definition of indifference class it follows that there are $a \in D_1 - D_2$, $b \in D_2 - D_1$, $c \in D_2 - D_3$ and $d \in D_3 - D_2$ such that $a > b : R_X$, $(bc) : R_X$ and $c > d : R_X$.
By the $\langle \bar{a}\ \overline{rs}\ \bar{a}, \bar{a}\rangle$-transitivity it follows that $a > d : R_X$.
Hence, by (2.6.3.6) $(D_1 - D_3) > (D_3 - D_1) : R_X$.

(2.6.3.5 c) Suppose $1 \le i < j \le k$. It is sufficient to prove that $C_i \cap C_j \subseteq C_t$ for all $i \le t \le j$.
For $i = t$ or $t = j$ this is trivial.
Suppose: $x \in C_i \cap C_j$ and $x \notin C_t$ for $i < t < j$.
We deduce a contradiction and are done.
$C_t \not\subseteq C_j$, since C_t is a maximal indifference class.
Hence, there exists a $y \in C_t - C_j$.
Since $t < j$ and $x \in C_j - C_t$, it follows by (2.6.3.5 b) that $y > x : R_X$.
Since $i < t$ and $x \in C_i - C_t$, it follows again by (2.6.3.5 b) that $y \notin C_t - C_i$. Hence, $y \in C_i$. This contradicts the fact that C_i is an indifference class of R_X.

(2.6.3.2) (only if) Since $S(U) \subset I(U)$, it is sufficient to prove (2.6.3.5 d).
Suppose $C_i \cap C_{i+2} \neq \varphi$ for some $1 \le i \le k - 2$.
It is straightforward to prove that $C_{i+1} \not\subseteq C_i \cup C_{i+2}$.
Hence, there exist $x_0 \in C_i - C_{i+1}$, $x_1 \in C_{i+1} - (C_i \cup C_{i+2})$, $x_2 \in C_{i+2} - C_{i+1}$ and $x_3 \in C_i \cap C_{i+2}$.
$\langle x_0, x_1, x_2, x_3\rangle$ is a path of type $\bar{a}^2\ \overline{rs}$, which cannot be cut short by a path of type $\bar{a}$. This contradicts the $\langle \bar{a}^2\ \overline{rs}, \bar{a}\rangle$-transitivity of R_X. Hence, $C_i \cap C_{i+2} = \varphi$.

(if) Evident and therefore left to the reader.

(2.6.3.3) and (2.6.3.4) are well-known.

■

We will define a sequence of extensions of W(U).
For all $k \geq 1$:

Let $\overline{rs}^k := \underbrace{\overline{rs}\ \overline{rs}\ \overline{rs}....\overline{rs}}_{k \text{ times}}$,

a path of type $\overline{rs}^0$ a path of length zero and

$S_k(U) := \{R_X \in Å$: R_X is reflexive, complete,
$\langle \overline{rs}\ \overline{a}\ \overline{a}\ \overline{rs}, \overline{a}\rangle$-classifiable transitive,
$\langle \overline{a}\ \overline{rs}\ \overline{a}, \overline{a}\rangle$-classifiable transitive and
$\langle \overline{rs}^k, \overline{sca}^{k-1}\rangle$-classifiable transitive$\}$.

Notice that $\langle \overline{rs}\ \overline{a}\ \overline{a}\ \overline{rs}, \overline{a}\rangle$-classifiable transitivity implies $\langle \overline{a}\ \overline{a}\ \overline{rs}, \overline{a}\rangle$-classifiable transitivity and $\langle \overline{rs}^k, \overline{sca}^{k-1}\rangle$-classifiable transitivity implies $\langle \overline{rs}^{k+1}, \overline{sca}^k\rangle$-classifiable transitivity.
Hence: $S_1(U) = L(U) \subset S_2(U) = W(U) \subseteq S_3(U) \subseteq S_4(U) \subseteq .... \subseteq S(U)$.
Evidently $S_k(U)$ is a set of relations, which can be classified as a set of orderings. We will now state a characteristic property for the relations in $S_k(U)$.

<u>Theorem 2.6.4</u>

Let $k \geq 1$. Then (2.6.4.1) and (2.6.4.2) are equivalent.

2.6.4.1 $R_X \in S_k(U)$.

2.6.4.2 There exist maximal indifference classes $C_1, C_2, ... C_t$ of R_X such that:

(a) $X = \cup \{C_i : i \in \{1,2,...,t\}\}$,

(b) for all $1 \leq i < j \leq t : C_i - C_j > C_j - C_i : R_X$,

(c) for all $1 \leq i < i+1 < j \leq t : C_i \cap C_j = \varphi$, and

(d) for all $1 \leq i < j \leq t$ and all $i \leq l < l+1 \leq j$, with $C_l \cap C_{l+1} \neq \varphi$:
$j - i \leq k - 1$ and $C_l \subseteq C_{l+1} \cup C_{l-1}$ for all $i < l < j$.

Proof of theorem 2.6.4

(2.6.4.1) -> (2.6.4.2) By (2.6.3.2) and $S_k(U) \subseteq S(U)$, it follows that (2.6.4.2)(a),(b) and (c) hold.

We will prove (2.4.6.2 d)

Suppose: $1 \leq i < j \leq t$ and $C_l \cap C_{l+1} \neq \varphi$ for all $i \leq l < j$.

Take $x_0 \in C_i - C_{i+1}$, $x_{l-i} \in C_l \cap C_{l+1}$ for all $i < l < j$ and $x_{j-i} \in C_j - C_{j-1}$.

Evidently, $\pi = \langle x_{j-1}, x_{j-2} \ldots, x_1, x_0 \rangle$ is a path of type $\overline{rs}^{j-i}$.

By (2.6.4 b) and (2.6.4 c) it follows that π cannot be cut short by a path of type $\overline{sca}^m$, such that $m < j - i$.

Hence, $j - i \leq k - 1$.

Suppose: $i < l < j$ and $C_l \not\subseteq C_{l-1} \cup C_{l+1}$.

Take $x_0 \in C_{l-1} \cap C_l$, $x_1 \in C_{l-1} - C_l$,
$x_2 \in C_l - (C_{l-1} \cup C_{l+1})$, $x_3 \in C_{l+1} - C_l$ and $x_4 \in C_{l+1} \cap C_l$.

Then $\langle x_0, x_1, x_2, x_3, x_4 \rangle$ is a path of type $\overline{rs}\ \bar{a}^2\ \overline{rs}$ which cannot be cut short by a path of type $\bar{a}$.

Hence, $C_l \subseteq C_{l-1} \cup C_{l+1}$.

(2.6.4.1) <- (2.4.6.2) By (2.6.3.2) (a), (b) and (c) it follows that $R_X \in S(U)$.

It is sufficient to prove that R_X is:

$\langle \overline{rs}^k, \overline{sca}^{k-1} \rangle$-transitive and $\langle \overline{rs}\ \bar{a}\ \bar{a}\ \overline{rs}, \bar{a} \rangle$-transitive.

<u>$\langle \overline{rs}^k, \overline{sca}^{k-1} \rangle$-transitivity</u> Let $\pi = \langle x_1, x_2, \ldots, x_{k+1} \rangle$ be a path along R_X of type $\overline{rs}^k$.

Then there are $D_1, D_2, \ldots, D_k \in \{C_1, C_2, \ldots, C_t\}$, such that $x_1, x_2 \in D_1$, $x_2, x_3 \in D_2 \ldots . x_k, x_{k+1} \in D_k$.

By (2.6.4.2) (d) and (c) there is a $D_i = D_j$ where $i < j$.

Hence, $\langle x_1, x_2, \ldots, x_i, x_{j+1}, \ldots, x_{k+1} \rangle$ is a short cut of type $\overline{rs}^m$ of π for some $m < k$, since R_Y is complete it is a short cut of type $\overline{sca}^m$.

Hence, R_X is $\langle \overline{rs}^k, \overline{sca}^{k-1} \rangle$-transitive.

<u>$\langle \overline{rs}\ \bar{a}^2\ \overline{rs}, \bar{a} \rangle$-transitivity</u> Let $\pi = \langle x_1, x_2, x_3, x_4, x_5 \rangle$ be a path along R_X of type $\overline{rs}\ \bar{a}^2\ \overline{rs}$. Since $R_X \in S(U)$ it is sufficient to prove $\langle x_1, x_5 \rangle \in \bar{a}R_X$ for the case $|\{x_1, x_2, x_3, x_4, x_5\}| = 5$.

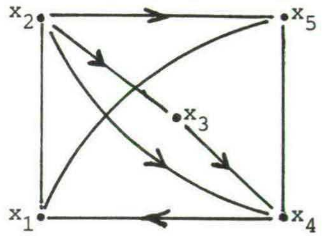
Note that $\langle x_2,x_4\rangle \in \bar{a}R_X$.

Hence, $\langle x_5,x_1\rangle \notin \bar{a}R_X$, since otherwise by the $\langle \bar{a}\ \overline{rs}\ \bar{a},\ \bar{a}\rangle$-transitivity of R_X it would follow that $\langle x_2,x_1\rangle \in \bar{a}R_X$.

Suppose $\langle x_1,x_5\rangle \in \overline{rs}R_X$.

By the $\langle \bar{a}\ \bar{a}\ \overline{rs},\ \bar{a}\rangle$-classifiable transitivity of R_X it follows that $\langle x_1,x_4\rangle \in \bar{a}R_X$ and $\langle x_2,x_5\rangle \in \bar{a}R_X$.

R_X:



Now there exist $D_1,D_2,D_3,D_4 \in \{C_1,C_2,\ldots,C_t\}$.
such that $x_1,x_2 \in D_1 = C_{i_1}$,

$x_1,x_5 \in D_2 = C_{i_2}$,

$x_4,x_5 \in D_3 = C_{i_3}$, and

$x_3 \in D_4 = C_{i_4}$.

Since $\langle x_2,x_5\rangle \in \bar{a}R$ and $x_1 \in D_1 \cap D_2$ it follows by (2.6.4.2) (b) and (c) $i_2 = i_1 + 1$. Similarly it follows $i_3 = i_2 + 1$. By (2.6.4.2 b) it follows $i_3 > i_4 > i_1$. Hence, $i_4 = i_2$. But $C_{i_2} \not\subseteq C_{i_1} \cup C_{i_3}$, since $x_3 \notin C_{i_1} \cup C_{i_3}$. Hence, (2.6.4.2 d) is violated. So $\langle x_1,x_5\rangle \in \bar{a}R_X$.

■

Theorem 2.6.4 leads to the following corollary:

<u>Corollary 2.6.5</u>

Let $W \subseteq Å$ be classifiable as a set of orderings.

Then (2.6.5.1) and (2.6.5.2) are equivalent.

2.6.5.1 $S_k(U) \subseteq W$.

2.6.5.2 There is a $\tilde{R}^k_X \in W$, such that

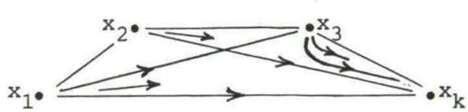
(a) $|X| = k$ and $X = \{x_1,x_2,\ldots,x_k\}$, and

(b) $\tilde{R}^k_X = \langle\{\langle x_i,x_j\rangle \in X \times X : i < j \text{ or } j+1 = i\}, X\rangle$.

■

Before proving this theorem we make a few remarks:

+ Corollary 2.6.5 is a generalization of theorem 2.3.14 and corollary 2.4.7.
+ Notice that $\bar{s}\tilde{R}^k_X = \langle\{\langle x_i, x_j\rangle \in X \times X : |i - j| \leq 1\}, X\rangle$.



$: \tilde{R}^k_X$

Hence, there exists a unique path $\pi = \langle x_1, x_2, x_3, \ldots, x_k\rangle$ of length k-1 and of type $\overline{rs}^{k-1}$ from x_1 to x_k along $\tilde{R}^k_X$ which cannot be cut short by a path of type $\overline{rs^{m}}$ where $m < k-1$. Furthermore, it is evident that $\tilde{R}^k_X \in S_k(U)$.

+ By corollary 2.6.5 it follows that $\tilde{R}^k_X$ carries all the information necessary to build $S_k(U)$.

<u>Proof of corollary 2.6.5</u>

(2.6.5.1) -> (2.6.5.2) Is trivial, since $\tilde{R}^k_X \in S_k(U)$.

(2.6.5.2) -> (2.5.6.1) Suppose $\tilde{R}^k_X \in W$.

By (2.4.3) we have $\Sigma_4\Sigma_6(L(U) \cup \{\tilde{R}^k_X, \bar{v}\tilde{R}^k_X\}) \subseteq W$.

It suffices to prove that $S_k(U) = \Sigma_4\Sigma_6(L(U) \cup \{\tilde{R}^k_X, \bar{v}\tilde{R}^k_X\})$.

It is straightforward, although cumbersome, to show that this follows from (2.6.4).

∎

Obviously by corollary 2.6.5 it follows that $S_{k+1}(U)$ is an extension of $S_k(U)$ for all $k \geq 1$. We will now prove that $S_{k+1}(U)$ is a minimal extension of $S_k(U)$. Hence, $S_3(U)$ is a minimal extension of W(U).

<u>Theorem 2.6.6</u>

For all $k \geq 1$, $S_{k+1}(U)$ is a minimal extension of $S_k(U)$.

<u>Proof of theorem 2.6.6</u>

Notice that for all $k \geq 2$:

$\tilde{R}^k_X \notin S_{k-1}(U)$ and for all $\varphi \neq Y \subset X$ it holds that

$\tilde{R}^k_X|_Y \in S_{k-1}(U)$. Hence, we are ready by (2.4.5), since evidently it holds that:

$$S_k(U) = \Sigma_4\Sigma_6(S_{k-1}(U) \cup \{\bar{v}\tilde{R}^k_X, \tilde{R}^k_X\})$$
$$= \Sigma_4\Sigma_6(L(U) \cup \{\bar{v}\tilde{R}^k_X, \tilde{R}^k_X\}).$$

■

Theorem 2.6.6 is a generalization of theorem 2.4.6. Furthermore, we observe that the infinite sequence $L(U) \subsetneq_m W(U) \subsetneq_m S_3(U) \subsetneq_m S_4(U) \subsetneq_m \ldots$ is a sequence of extensions such that for all $k \geq 1$ the transitivity property is slightly weakened, when going from $S_k(U)$ to $S_{k+1}(U)$. Since $S_{k+1}(U)$ is a minimal extension one might say that this weakening is minimal.

Before examining minimal extension of S(U) and I(U), we will try to solve the following interesting problem:

Find a collection C of minimal extensions of W(U), such that for all minimal extensions of W(U) it holds that it is isomorphic to precisely one of the members of the collection. (*)

<u>Theorem 2.6.7</u>

Let $W \subseteq Å$ be classified as a set of orderings.
If W is an extension of W(U), then there exists a set W' isomorphic to W such that $W(U) \subset W'$.

<u>Proof of Theorem 2.6.7</u>

Suppose W is an extension of W(U).
Then there exists a set $W'' \subset W$, such that $W(U) \simeq W''$.

By corollary 2.2.27 $W'' \in \{W(U), W(U)^{\overline{cv}}, W(U)^{\bar{q}}, W(U)^{\overline{qcv}}\}$.

So $W(U) \subset W$, $W(U) \subset W^{\overline{cv}}$, $W(U) \subseteq W^{\bar{q}}$ or $W(U) \subseteq W^{\overline{qcv}}$.

Hence, there exists a set W' in $\{W, W^{\overline{cv}}, W^{\bar{q}}, W^{\overline{qcv}}\}$ such that $W(U) \subset W'$. Furthermore by (2.2.27), $W \simeq W'$.

■

So, in order to solve (*), by theorem 2.6.7 it is sufficient to find extensions W of W(U) such that $W(U) \subset W$. Let us first define $A_3(U)$, $C_3(U)$ and $B_2(U)$, and then prove that these are minimal extensions of W(U) and moreover that they form a collection we are looking for.

Of course $S_3(U)$ is a minimal extension of $W(U)$.

Let:

$A_3(U) := \{R_X \in \mathring{A}$: R_X is strongly complete,
R_X is $\langle\overline{rs}\ \overline{rs},\overline{sca}\rangle$-classifiable transitive and
R_X is $\langle\bar{a}\ \overline{rs}\ \bar{a}\ \overline{rs}\ \bar{a},\bar{a}\rangle$-classifiable transitive$\}$,

$C_3(U) := \{R_X \in \mathring{A}$: R_X is strongly complete,
R_X is $\langle\overline{rs}\ \overline{rs},\overline{sca}\rangle$-classifiable transitive and
R_X is $\langle\overline{rs}\ \bar{a}\ \overline{rs}\ \bar{a}\ \overline{rs}\ \bar{a}\ \overline{rs},\overline{cav}\rangle$-classifiable transitive$\}$, and

$B_2(U) := \{R_X \in \mathring{A}$: R_X is reflexive &
R_X is $\langle\overline{cav}\ \overline{cav},\overline{cav}\rangle$-classifiable transitive$\}$.

Of course $B_2(U) \subset V_1(U)$.

We are going to establish the following picture:

$$W(U) \begin{cases} \underset{m}{\subset} B_2(U) \\ \underset{m}{\subset} A_3(U) \subset A(U) \\ \underset{m}{\subset} C_3(U) \subset C(U) \\ \underset{m}{\subset} S_3(U). \end{cases}$$

Evidently $S_3(U)$, $A_3(U)$, $C_3(U)$ and $B_2(U)$ are extensions of $W(U)$. We will now prove that $A_3(U)$, $C_3(U)$ and $B_2(U)$ are minimal extensions of $W(U)$.

We start with proving that $B_2(U)$ is a minimal extension of $W(U)$.

<u>Theorem 2.6.8</u>

$B_2(U)$ is a minimal extension of $W(U)$.

<u>Proof of theorem 2.6.8</u>

It suffices to prove that $B_2(U) = \Sigma_4\Sigma_6(W(U) \cup \{\bar{v}R_X,R_X\})$, where $\bar{v}R_X = R_X = Id_X$ and $X \in$ Æ and $|X| = 2$.
Note that $R_X \in B_2(U)$. Hence, it follows that
$\Sigma_4\Sigma_6(W(U) \cup \{\bar{v}R_X,R_X\}) \subseteq B_2(U)$.
We will prove that $B_2(U) \subseteq \Sigma_4\Sigma_6(W(U) \cup \{\bar{v}R_X,R_X\})$ and we are done.

Let $R_Y \in B_2(U)$. We have to show that
$R_Y \in \Sigma_4\Sigma_6(W(U) \cup \{\bar{v}R_X, R_X\})$.
Let $Y_1, Y_2, \ldots, Y_n$ be a partition of Y such that Y_i is a <u>maximal reversible class</u> of R_Y, where D is a <u>reversible class</u> of R_Y iff $R_{Y|_D} = \bar{v}R_{Y|_D}$, and D is a <u>maximal reversible class</u> of R_Y iff D is a reversible class of R_Y and for all reversible classes D' of R_Y : $D \not\subset D'$.
It is straightforward to prove that these classes exist because of the $\langle\overline{cav}\ \overline{cav},\overline{cav}\rangle$-transitivity of R_Y.
Furthermore, by this transitivity it follows that
$R_Y = R_{Y_1} \gg R_{Y_2} \gg \ldots \gg R_{Y_k}$. (for some enumeration of these classes).
Since by the definition of Σ_6 it holds that
$R_{Y_i} \in \Sigma_6(W(U) \cup \{R_X\})$ for all $i \in \{1,\ldots,k\}$, we are done.

■

We will now prove that $A_3(U)$ and $C_3(U)$ are minimal extensions of W(U).

Let $C(U) = \{R_X \in \mathring{A} : R_X$ is strongly complete and

R_X is $\langle\overline{rs}\ \overline{rs},\overline{sca}\rangle$-transitive$\}$.

Of course C(U) can be classified as a set of orderings. $A_3(U)$, W(U) and $C_3(U)$ are all subsets of C(U). Going from W(U) to Q(U) the $\langle\overline{rs}\ \overline{rs},\overline{rs}\rangle$-transitivity is dropped and going from W(U) to C(U) the $\langle\bar{a}\ \bar{a},\bar{a}\rangle$-transitivity is dropped.

We have the following lemma:

<u>Lemma 2.6.9</u>

Let $R_X \in C(U)$.
Then there exists a partition $C_1, C_2 \ldots, C_t$ of X such that for all i, C_i is a maximal indifference class of R_X.

<u>Proof of lemma 2.6.9</u>

Evident, since because of the $\langle\overline{rs}\ \overline{rs},\overline{sca}\rangle$-transitivity $\overline{rs}R_X$ is an equivalence relation.

■

We will now characterize the relations in $A_3(U)$. By the $\langle\bar{a}\,\overline{rs}\,\bar{a}\,\overline{rs}\,\bar{a},\bar{a}\rangle$-classifiable transitivity it follows that every relation in $A_3(U)$ is acyclic. Hence, $A_3(U)$ is a subset of the set of acyclical relations defined as follows:

$A(U) := \{R_X \in \mathring{A} : R_X$ is strongly complete and for all $t \geq 3$,
R_X is $\langle\bar{a}^t,\bar{n}\rangle$-classifiable transitive$\}$.

Note that R_X is $\langle\bar{a}^t,\bar{n}\rangle$-classifiable transitive, iff there is no cycle along R_X of type $\bar{a}^t$. $A(U)$ is the set of strongly complete and <u>acyclic</u> relations. In literature $\overline{A}^q(U)$ and $\overline{A}^{cv}(U)$ are sometimes called the set of acyclic relations. Corollary 2.2.27 clarifies again why these disjoint sets have the same name.

Furthermore, notice that $A(U)$ can be classified as a set of orderings, because of theorem 2.2.28 and theorem 2.3.12. Of course $A_3(U) \subset A(U)$.

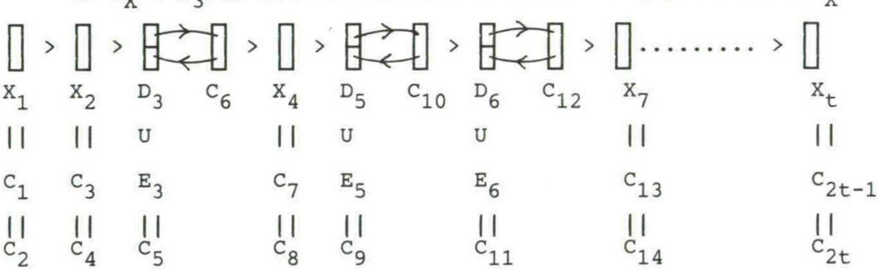
<u>Lemma 2.6.10</u>

Let $R_X \in \mathring{A}$. Then (2.6.10.1) and (2.6.10.2) are equivalent.

2.6.10.1 $R_X \in A_3(U)$.

2.6.10.2 There are maximal indifference classes $C_1,C_2...C_{2t}$ of R_X and $X_i = C_{2i} \cup C_{2i-1}$ for all $t \geq i \geq 1$, such that
- (a) $X_1,X_2...X_t$ is a partition of X,
- (b) for all $i < j$, $X_i > X_j : R_X$, and
- (c) for all X_i, with $X_i \neq C_{2i}$, there is a partition D_i, E_i of C_{2i-1}, such that $D_i > C_{2i} : R_X$ and $C_{2i} > E_i : R_X$. ■

So $R_X \in A_3(U)$ iff we can make the following picture of R_X:



Hence, $R_X \in A_3(U)$ if and only if we can order its maximal indifference classes linearly except some local disjoint distortions, which are expressed by II(c).

<u>Proof of lemma 2.6.10</u>

(2.6.10.1) $\rightarrow$ (2.6.10.2) Suppose $R_X \in A_3(U)$.

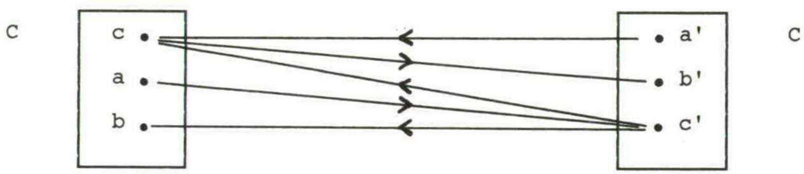
By lemma 2.6.9 there are maximal indifference classes $C_1, C_2, \ldots, C_k$ of R_X, which is a partition of X. Let C,C' be different maximal indifference classes of R_X.

2.6.10.1.1 There exist B, B' such that $\{B, B'\} = \{C, C'\}$ and either $B > B' : R_X$ or there exist $D, E \subset B$ such that $D \cup E = B$, $D \cap E = \varphi$, $D > B' : R_X$ and $B' > E : R_X$.

<u>Proof of (2.6.10.1.1)</u>

It is sufficient to prove that the following assumption leads to a contradiction: there are $a,b,c \in C$ and $a',b',c' \in C'$, with $\{\langle a',c\rangle, \langle a,c'\rangle, \langle c,b'\rangle, \langle c',b\rangle\} \subseteq \bar{a}R_X$.

Since $c' \notin C$ it follows by lemma 2.6.9 that $\langle c,c'\rangle \in \bar{a}R_X$ or $\langle c',c\rangle \in \bar{a}R_X$. Because of reasons of symmetry suppose without loss of generality $\langle c',c\rangle \in \bar{a}R_X$.



Now $\langle c, b', c', b, a, c'\rangle$ is a path of type $\bar{a}\ \overline{rs}\ \bar{a}\ \overline{rs}\ \bar{a}$.

Hence, by the $\langle \bar{a}\ \overline{rs}\ \bar{a}\ \overline{rs}\ \bar{a}, \bar{a}\rangle$-transitivity of R_X it holds that $\langle c,c'\rangle \in \bar{a}R_X$, which contradicts $\langle c',c\rangle \in \bar{a}R_X$.

Using again the $\langle \bar{a}\ \overline{rs}\ \bar{a}\ \overline{rs}\ \bar{a}, \bar{a}\rangle$-transitivity of R_X it follows immediately that there are no circuits along R_X of type $\bar{a}\ \overline{rs}\ \bar{a}\ \overline{rs}\ \bar{a}\ \overline{rs}$. From this and (2.6.10.1.1) it follows that the maximal indifference classes of R_X can be ordered as indicated in (2.6.10.2).

(2.6.10.2) $\rightarrow$ (2.6.10.1) Is straightforward and therefore omitted.

■

Now we have as a consequence of (2,6,10) and (2.4.5)

Corollary 2.6.11

$A_3(U) = \Sigma_4\Sigma_6(W(U) \cup \{\bar{v}R_Y, R_Y\})$ is a minimal extension of $W(U)$, where $Y = \{a,b,c\}$, $|Y| = 3$ and

$R_Y = \overline{cav}<\{<b,a>,<c,b>\},Y>$.

■

We will now prove that $C_3(U)$ is a minimal extension of $W(U)$. First we will characterize the relations $C_3(U)$.

Lemma 2.6.12

Let $R_X \in Å$. Then (2.6.12.1) and (2.6.12.2) are equivalent.

2.6.12.1 $R_X \in C_3(U)$.

2.6.12.2 There is a partion $C_1, C_2 \ldots C_t$ of X, which members are maximal indifference classes of R_X, and there are $d_i \in C_i$, for all $i \in \{1,2\ldots t\}$, such that:

(a) $C_i > C_j : R_X$ iff $d_i > d_j : R_X$, for all $i,j \in \{1,2\ldots t\}$, and

(b) $R_{X|_D} \in T_3(U)$, where $D := \{d_1,d_2\ldots d_t\}$.

■

Lemma 2.6.12 states that $R_X \in C_3(U)$ iff its maximal indifference classes can be ordered as a tournament of $T_3(U)$.

Proof of lemma 2.6.12

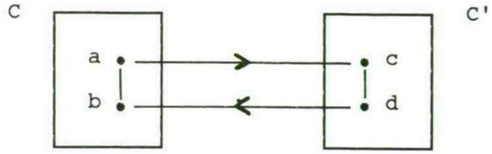
(2.6.12.1) -> (2.6.12.2) Let $C_1, C_2 \ldots C_t$ be the maximal indifference classes of R_X.

By lemma 2.6.9 $C_1, C_2 \ldots C_t$ is a partition of X.

Take $d_1 \in C_1$, $d_2 \in C_2 \ldots d_t \in C_t$ arbitrarily.

Let $D = \{d_1, d_2 \ldots d_t\}$.

(2.6.12.2 a) Let $\{C, C'\} \subseteq \{C_1, C_2 \ldots C_t\}$. It is sufficient to prove that the following assumption leads to a contradiction: there are $a,b \in C$ and $c,d \in C'$, such that $\{<a,c>,<d,b>\} \subseteq \bar{a}R_X$.



Note that $\langle b, a, c, d, b, a, c, d\rangle$ is a path along R_X of type $\overline{rs}\ \bar{a}\ \overline{rs}\ \bar{a}\ \overline{rs}\ \bar{a}\ \overline{rs}$. Now by $\langle\overline{rs}\ \bar{a}\ \overline{rs}\ \bar{a}\ \overline{rs}\ \bar{a}\ \overline{rs}, \overline{cav}\rangle$-transitivity of R_X, it follows that $\langle b,d\rangle \in \overline{cav}R_X$.

This contradicts $\langle d,b\rangle \in \bar{a}R_X$.

(2.6.12.2 b) Note that $R_{X|_D} \in T(U)$ by (2.6.12.2 a).

Furthermore, by the $\langle\overline{rs}\ \bar{a}\ \overline{rs}\ \bar{a}\ \overline{rs}\ \bar{a}\ \overline{rs}, \overline{cav}\rangle$-transitivity of R_X it follows that there are no circuits of length 4 along $R_{X|_D}$.

Using the theory developed in § 2.5, it follows that $R_{X|_D} \in T_3(U)$.

(2.6.12.2) -> (2.6.12.1) is straightforward and therefore omitted.

■

By Lemma 2.6.12 and theorem 2.4.5 we have

<u>Corollry 2.6.13</u>

$C_3(U) = \Sigma_4\Sigma_6(W(U) \cup \{R_Y\})$ is a minimal extension of $W(U)$, where $Y = \{a,b,c\}$, $|Y| = 3$ and

$R_Y = \overline{cav}\langle\{\langle a,c\rangle,\langle c,b\rangle,\langle b,a\rangle\},Y\rangle$.

■

Now we are ready to prove that $B_2(U)$, $S_3(U)$, $C_3(U)$ and $A_3(U)$ are the minimal extensions of $W(U)$, disregarding sets of relations, which are isomorphic to one of them.

Theorem 2.6.14

Let $W \subseteq Å$ be classified as a set of orderings.

Then the following holds:

if W is a minimal extension of W(U), then W is isomorphic to either $B_2(U)$ or $S_3(U)$ or $C_3(U)$ or $A_3(U)$.

Proof of theorem 2.6.14

Suppose W is a minimal extension of W(U) and W is classified as a set of orderings.

By theorem 2.6.7 there exists a $W' \cong W$, such that $W(U) \subset W' \subseteq Å$.

Since W is a minimal extension of W(U) it follows evidently that W' is a minimal extension of W(U).

By (2.4.5) it follows that $W' = \Sigma_4\Sigma_6(W(U) \cup \{R_B, \bar{v}R_B\})$ for some $R_B \in W' - W(U)$, such that for all $C \subset B$, with $C \neq \varphi$, $R_{B|_C} \in W(U)$.

It is straightforward to calculate that: $R_B \in \{R_X^1, R_Y^2, R_Y^3, R_Y^4\}$, where $Y = \{a,b,c\}$, $|X| = 2$, $|Y| = 3$,

$R_X^1 = \bar{r}\varphi_X$,

$R_Y^2 = \overline{cav}\langle\{\langle b,a\rangle\}, Y\rangle$,

$R_Y^3 = \overline{cav}\langle\{\langle b,a\rangle, \langle c,b\rangle\}, Y\rangle$, and

$R_Y^4 = \overline{cav}\langle\{\langle b,a\rangle, \langle c,b\rangle, \langle a,c\rangle\}, Y\rangle$.

By theorem 2.6.8 $B_2(U) = \Sigma_4\Sigma_6(W(U) \cup \{R_X^1\})$.

By corollary 2.6.13 $C_3(U) = \Sigma_4\Sigma_6(W(U) \cup \{R_Y^4, \bar{v}R_Y^4\})$.

By corollary 2.6.11 $A_3(U) = \Sigma_4\Sigma_6(W(U) \cup \{R_Y^3, \bar{v}R_Y^3\})$.

By theorem 2.6.4 $S_3(U) = \Sigma_4\Sigma_6(W(U) \cup \{R_Y^2, \bar{v}R_Y^2\})$.

We are done if we can prove that none of the sets $S_3(U)$, $A_3(U)$, $C_3(U)$ and $B_2(U)$ are isomorphic to each other. This follows immediately from corollary 2.2.27 and the definitions of these sets.

■

In the preceding pages we have investigated the minimal extensions of W(U). We found four ways to extend the set of weak orderings:

(A) by dropping the completeness condition, we found $B_2(U)$,

(B) by weakening the transitivity condition on the symmetric part of the relation we found $S_3(U)$,

(C) by weakening the transitivity condition on the asymmetric part to a condition of acyclicity we found $A_3(U)$, and
(D) by weakening the transitivity condition of the asymmetric part by admitting circuits of type $\bar{a}^3$ we found $C_3(U)$.

(B), (C) and (D) appear to be three minimal weakenings of the transitivity condition.

Notice that the minimality of the extension W(U) of L(U) and of the extensions $B_2(U)$, $S_3(U)$, $A_3(U)$ and $C_3(U)$ of W(U) has been shown by the following type of characterization of the extension W: if V can be classified as a set of orderings, then $W \subseteq V$ iff a certain R_X is in V.

So, that particular relation R_X has all the information to build all the relations in W. Hence, it is evident that $|X|=2$ for the case where $W \in \{W(U), B_2(U)\}$ and $|X|=3$ for the case where $W \in \{S_3(U), A_3(U), C_3(U)\}$, since in the latter case there is a weakening of the transitivity conditions (hence at least three elements should be involved) and in the former case there is a weakening of the completeness and antisymmetry.

Next we will investigate minimal extensions of S(U) and I(U) (See the picture at the end of this section). This investigation is restricted to extensions, which are subsets of Q(U), since we want to preserve the transitivity of the antisymmetric part of the relations.

To begin with we define two sequences of extensions of S(U).

<u>Definition 2.6.15</u> <u>Sequences of extensions of S(U)</u>

Let $k \geq 1$.

$I'_k(U) := \{R_X \in \AA :$ There exists a set of maximal indifference classes of R_X, say $\{C_1, C_2 \ldots C_t\}$ such that (2.6.15.1), (2.6.15.2) and (2.6.15.3) hold$\}$,

$I_k(U) := \{R_X \in \AA :$ There exists a set of maximal indifference classes of R_X, say $\{C_1, C_2 \ldots C_t\}$ such that (2.6.15.1), (2.6.15.2), (2.6.15.3) and (2.6.15.4) hold$\}$, where:

2.6.15.1 $\cup \{C_i : i \in \{1,2 \ldots t\}\} = X$,

2.6.15.2 for all $1 \leq i < j \leq t : (C_i - C_j) > (C_j - C_i) : R_X$, 2.6.15.3 for all $1 \le i \le t-k$: $C_i \cap C_{i+k} = \varphi$, and

2.6.15.4 for all $1 \le i \le t-k+1$, with $C_i \cap C_{i+k} \ne \varphi$, it holds:

(a) for all $1 \le n \le i$: $C_i \cap C_n = \varphi$,

(b) for all $i+k-1 < n \le t$: $C_{i+k-1} \cap C_n = \varphi$, and

(c) for all $i \le n < m \le i+k-1$: $C_n \cap C_m = C_i \cap C_{i+k-1}$.

■

It is straightforward, although cumbersome, to prove that $I_k(U)$ and $I'_k(U)$ can be classified as sets of orderings for all $k \ge 1$. Furthermore, the following holds evidently:

$I_1(U) = I'_1(U) = W(U) \subset I_2(U) = S_3(U) \subset I'_2(U) = S(U) \subset I_3(U) \ldots$
$\ldots \subset I'_k(U) \subset I_{k+1}(U) \subset I'_{k+1}(U) \ldots \subset I(U)$.

In the following theorems it will be proved that $I_{k+1}(U)$ is a minimal extension of $I'_k(U)$, in a similar way as was shown that $W(U)$ is a minimal extension of $L(U)$ and $S_3(U)$, $A_3(U)$, $B_2(U)$ and $C_3(U)$ are minimal extensions of $W(U)$.

<u>Theorem 2.6.16</u>

Let $k \ge 1$. Let $W \subset \mathring{A}$ be classified as a set of orderings, such that $I'_k(U) \subseteq W \subseteq I(U)$ and furthermore let $\hat{R}_{X_k} \in \mathring{A}$ be defined as follows: $|X_k| = k$, $X_k = \{x_1, x_2, \ldots, x_k\}$ and $\hat{R}_{X_k} := \langle\{\langle x_i, x_j\rangle : (1 \le i \le j \le k) \vee (i=k \;\&\; 1 \le j \le k)\}, X_k\rangle$.

Then the following holds:

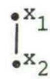
$\hat{R}_{X_k} \in W$ iff $I_k(U) \subseteq W$.

■

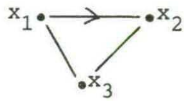
Before proving theorem 2.6.16, notice that $\hat{R}_{X_k}|_{\{x_1, x_2, \ldots, x_{k-1}\}} \in L(U)$ and x_k is indifferent to all the other elements of X_k in R_{X_k}. Hence, the set of maximal indifference classes of $\hat{R}_{X_k}$ is as follows: $\{\{x_1,x_k\},\{x_2,x_k\},\ldots,\{x_{k-1},x_k\}\}$. Let $C_i = \{x_i,x_k\}$. Observe that $\cup\{C_i : 1 \le i \le t\} = X_k$ and $(C_i - C_j) > (C_j - C_i) : \hat{R}_{X_k}$ for all $1 \le i < j \le k$. Hence obviously $\hat{R}_{X_k} \in I_k(U) - I'_{k-1}(U)$. In the proof of theorem 2.6.16 will be shown that $\hat{R}_{X_k}$ "carries" all the information to build every element of $I_k(U) - I'_{k-1}(U)$ for $k \ge 2$. To give the reader a little bit more insight into the relations $\hat{R}_{X_k}$, we draw a few diagrams of these relations.

$\hat{R}_{X_1}$:

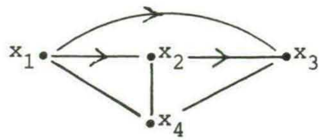
$\hat{R}_{X_2}$:



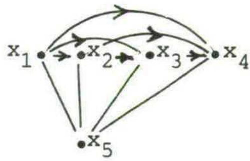
$\hat{R}_{X_3}$:



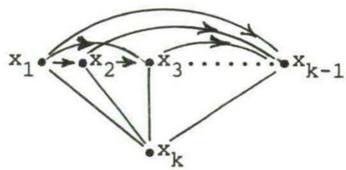
$\hat{R}_{X_4}$:



$\hat{R}_{X_5}$:



$\hat{R}_{X_k}$:



<u>Proof of theorem 2.6.16</u>

It is straightforward but cumbersome to prove that:

$$I_k(U) = \Sigma_4\Sigma_6(W(U) \cup \{\bar{v}\hat{R}_{X_k}, \hat{R}_{X_k}\})$$

$$= \Sigma_4\Sigma_6(I'_{k-1}(U) \cup \{\bar{v}\hat{R}_{X_k}, \hat{R}_{X_k}\}).$$

(only if) Suppose $\hat{R}_{X_k} \in W$. We have to prove that $I_k(U) \subseteq W$.

Since $\hat{R}_{X_k} \in W$, $W(U) \subseteq W$ and since W can be classified as a set of orderings, we have $\Sigma_4\Sigma_6(W(U) \cup \{\bar{v}\hat{R}_{X_k}, \hat{R}_{X_k}\}) \subseteq W$.

Hence, $I_k(U) \subseteq W$.

(if) Suppose $I_k(U) \subseteq W$. We have prove that $\hat{R}_{X_k} \in W$.

This is evident since $\hat{R}_{X_k} \in I_k(U)$.

■

<u>Corollary 2.6.17</u>

Let $k \geq 1$. Let $W \in Å$ be classified as a set of orderings, such that $W \subseteq I(U)$. Then the following holds:

W is a minimal extension of $I'_k(U)$ iff $I_{k+1}(U) = W$.

Proof of corollary 2.6.17

Notice that for all $\varphi \neq Y \subset X_k$, $\hat{R}_{X_k|_Y} \in I'_k(U)$.

Hence, since $\Sigma_4\Sigma_6(I'_k(U) \cup \{\hat{R}_{X_k}, \bar{v}\hat{R}_{X_k}\}) = I_{k+1}(U)$ by (2.4.5) it follows that $I_{k+1}(U)$ is a minimal extension of $I'_k(U)$. Hence, it suffices to prove: if $I'_k(U) \subset W$, then $\hat{R}_{X_{k+1}} \in W$. Suppose $I'_k(U) \subset W$. Then there exists a $R_Y \in W - I'_k(U)$ such that $R_Y \in I(U)$. Let $D_1, D_2, \ldots, D_m$ be the maximal indifference classes of R_Y. According to (2.6.3.1) (after a renumbering) it holds that:

(a) $\cup \{D_t : 1 \leq t \leq m\} = Y$,

(b) for all $1 \leq i < j \leq m : D_i - D_j > D_j - D_i : R_Y$, and

(c) for all $1 \leq i < j \leq m : D_i \cap D_j = \cap \{D_t : i \leq t \leq j\}$.

Since $R_Y \notin I'_k(U)$ it follows that there is a $n \in \{1,\ldots,m\}$, such that $D_n \cap D_{n+k} \neq \varphi$.

Take $d_i \in D_{n+i} - Z$ and $d_{k+1} \in Z$, where $Z = \cap \{D_t : n \leq t \leq n+k\}$.

Obviously $R_Y|_{\{d_1,\ldots,d_{k+1}\}} \in \Sigma_1(\{\hat{R}_{X_{k+1}}, \bar{v}\hat{R}_{X_{k+1}}\})$.

Hence, $\hat{R}_{X_{k+1}} \in W$.

■

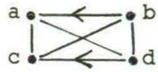
We end this section with investigating a minimal extension of I(U). This extension will be defined by means of conditions of a restrictive nature on the maximal indifferent classes of the relations in that extension:

$J(U) := \{R_X \in \text{Å}$: there exists a partion of X, say $Y_1, Y_2, \ldots Y_t$, such that: $R_X = (R_{X|_{Y_1}}) \gg (R_{X|_{Y_2}}) \gg \ldots \gg (R_{X|_{Y_t}})$ and for all $i \in \{1,\ldots,t\}$, with $R_{X|_{Y_i}} \notin I(U)$, there is a partition of Y_i say, D^1, D^2, D^3, D^4, such that

$$R_{X|_{Y_i}} = \bar{c}\langle\{\langle x,y\rangle : [x \in D^2 \ \&\ y \in D^1] \vee [x \in D_i^4 \ \&\ y \in D_i^3]\}, Y_i\rangle \ \}.$$

Let $\tilde{R}_A := \bar{c}\langle\{\langle b,a\rangle,\langle d,c\rangle\},A\rangle$, where $A = \{a,b,c,d\}$ and $|A| = 4$.

Hence, R_A:



Then it is obvious that $\tilde{R}_A \notin I(U)$, but for all $\varphi = B \subset A$ $\tilde{R}_{A|_B} \in I(U)$.

Moreover, it is straightforward to prove that: $\Sigma_4\Sigma_6(I(U) \cup \{\bar{v}\tilde{R}_A,\tilde{R}_A\}) = J(U)$. Hence, by (2.4.5), $J(U)$ is a minimal extension of $I(U)$.

We will prove that $J(U)$ is the "only" minimal extension of $I(U)$ which is in $Q(U)$. To prove this we make use of the following theorem.

Theorem 2.6.18

Let $W \subseteq Å$ be classified as a set of orderings, such that $I(U) \subseteq W \subseteq Q(U)$. Then the following holds:
$\tilde{R}_A \in W$ iff $J(U) \subseteq W$.

Proof of theorem 2.6.18

(if) Trivial, since $\tilde{R}_A \in J(U)$.

(only if) Suppose $\tilde{R}_A \in W$.

Obviously it holds that: $J(U) = \Sigma_4\Sigma_6(I(U) \cup \{\tilde{R}_A,\bar{v}\tilde{R}_A\}) \subseteq W$. ■

Next we will prove that $J(U)$ is the "only" minimal extension of $I(U)$, which is in $Q(U)$. The idea underlying the proof is from Fishburn [1970], who has shown that $R_X \in I(U)$ iff $\tilde{R}_A \neq \sigma(R_{X|_Y})$ for all $Y \subseteq X$ and $\sigma \in S_U$.

Corollary 2.6.19

Let $W \subseteq Å$ be classified as a set of orderings.
Then the following holds: $J(U)=W$ iff
$I(U) \subset W \subseteq Q(U)$ and W is a minimal extension of $I(U)$.

<u>Proof of corollary 2.6.19</u>

We have already shown that (if) holds.

(only if) It suffices to prove: if $I(\underset{\sim}{U}) \subset W \subseteq Q(U)$, and W is a minimal estension of $I(U)$, then $R_A \in W$.

Suppose: $I(U) \subset W \subseteq Q(U)$ and $R_A \notin W$. Then it follows for all $R_X \in W$ and $x_1, x_2, x_3, x_4 \in X$:

$[\langle x_1, x_2\rangle \in \bar{a}R_X \ \& \ \langle x_3, x_4\rangle \in \bar{a}R_X] \rightarrow$
$[\langle x_1, x_4\rangle \in \bar{a}R_X \vee \langle x_3, x_2\rangle \in \bar{a}R_X]$.

Hence, R_X is $\langle \bar{a}\ \overline{rs}\ \bar{a}, \bar{a}\rangle$-classifiable transitive and $R_X \in I(U)$. So $W \subseteq I(U)$. This contradicts the assumption $I(U) \subset W$, which completes the proof.

■

We conclude this section with an inclusion diagram of the sets of relations, which were found earlier and which can be classified as sets of orderings.

$$L(U)\begin{cases} \underset{m}{\subsetneq} T_3(U) \underset{m}{\subsetneq} T_4(U) \underset{m}{\subsetneq} T_{5,1,1}(U) \subset T_{5,2}(U) \subseteq T_5(U) \ldots\ldots \\ \qquad\qquad \ldots\ldots T_{k,1}(U) \underset{m}{\subsetneq} T_{k,1,1}(U) \subseteq \ldots T(U). \\ \underset{m}{\subsetneq} W(U) \begin{cases} \underset{m}{\subsetneq} B_2(U) \\ \underset{m}{\subsetneq} A_3(U) \subset A(U) \\ \underset{m}{\subsetneq} C_3(U) \subset C(U) \\ \underset{m}{\subsetneq} S_3(U) \underset{m}{\subsetneq} S_4(U) \ldots \subset S_k(U) \underset{m}{\subsetneq} S_{k+1}(U) \ldots \subset S(U) \underset{m}{\subsetneq} I_3(U) \end{cases} \end{cases}$$

$$S_3(U) = I_2(U), \qquad S(U) = I_2'(U)$$

$$\subset \ldots \subset I_k'(U) \underset{m}{\subsetneq} I_{k+1}(U) \ldots \subset I(U) \underset{m}{\subsetneq} J(U) \subset Q(U)$$

§ 2.7 Some remarks on sets of orderings

In this section we will try to recover some intuitive ideas concerning the sets of relations which can be classified as sets of orderings according to definition 2.2.22 and hope to give more insight into this classification system.

Observe that by theorem 2.3.14 it follows that, whenever W is a classifiable set of orderings, then W contains all those relations which order the elements in a non-branching chain from better to worse. So in those relations the elements can be put on a line starting with the best and ending with the worst element. Therefore the orderings are called linear orderings. L(U) is by definition the set of all linear orderings. Furthermore, by (2.3.14) it follows that there is no classifiable set of orderings which is a non-trivial subset of L(U). Hence, L(U) is the smallest classifiable set of orderings. A natural question is now whether there is a largest classifiable set. Of course, the answer is positive: take $V_1(U) := \{ R_X \in \AA: R_X \text{ is reflexive}\}$.

By theorem 2.2.23 it is easy to prove that for all $W \subset \AA$, which can be classified as a set of orderings, the following holds: $W \subseteq V_1(U)$ or $W \subseteq V_1(U)^{\bar{q}}$. Hence, it follows that $V_1(U)$ and $V_1(U)^{\bar{q}}$ are the largest classifiable sets of orderings. Since by (2.2.26) $V_1(U) \simeq V_1(U)^{\bar{q}}$, we can speak of <u>the</u> largest classifiable set of orderings.

Various minimal extensions introduced in §2.4, §2.5 and §2.6 corresponded with a minimal weakening of the transitivity condition of orderings. An interesting problem is the following:

Does there exist a set $W \subseteq \AA$, which is classifiable as a set of orderings such that $W \underset{m}{\subsetneq} V_1(U)$. Since by theorem 2.2.23 either all relations in a set which can be classified as a set of orderings, are reflexive or all these relations are irreflexive, it follows that the relations in $V_1(U)$ do not satisfy any transitivity condition. Hence, the problem stated above can be reformulated as the following question: Does a "minimal" condition exist? The answer to this question is given by the following theorem.

Theorem 2.7.1

Let $V \subset V_1(U)$ be a classifiable set of orderings. Then there exists a classifiable set of orderings W with $V \subset W \subset V_1(U)$.

Proof of theorem 2.7.1

Suppose $V \subset V_1(U)$.

Since $V \subset V_1(U)$ there exists a relation $R_X \in V_1(U)$, such that $R_X \notin V$. By the finiteness of X and simple induction reasonings we may assume without loss of generality that:

for all $Y \subset X$, with $Y \neq \varphi$, $R_{X|_Y} \in V$. (2.7.1.1)

By theorem 2.4.5, $\Phi(V \cup \{R_X\}) = \Sigma_4\Sigma_6(V \cup \{R_X, \bar{v}R_X\})$ is a minimal extension of V. (2.7.1.2)

It is sufficient to prove that $W \neq V_1(U)$, where $W = \Phi(V \cup \{R_X\})$.

Let $A = X \cup \{a\}$, where $a \notin X$ and $a \in U$.

Now let $X = \{x_1,x_2,x_3,\ldots,x_k\}$ such that for all $i,j \in \{1,2,\ldots,k\}$ it holds that:

$i \geq j$ iff $|\{x \in X: \langle x_i,x\rangle \in \bar{a}R_X\}| \leq |\{x \in X: \langle x_j,x\rangle \in \bar{a}R_X\}|$.

Define $R_A := \langle\{\langle x,y\rangle \in A \times A : [\langle x,y\rangle \in R_X] \vee$
$[x = x_i \ \& \ y = a \ \& \ i \in \{2,3,..,k\}] \vee$
$[x = a \ \& \ y = x_1]\},A\rangle$.

It is sufficient to prove that $R_A \notin W$.

Suppose $R_A \in W$.

If R_A is reducible (i.e., $R_A = R_B \gg R_C$) then R_X is reducible. So $R_X = R_Y \gg R_Z$ and by (2.7.1.1) it follows $R_X \in W$, which is not the case.

Therefore, by (2.7.1.2) we have that $R_A \in \Sigma_6(V \cup \{R_X, \bar{v}R_X\})$.

Hence, there is a partion $B_1,B_2,\ldots,B_m$ of A and there are relations $R_{B_1}, R_{B_2},\ldots,R_{B_m} \in \Omega(W)$ (See 2.2.20) and $R_C \in V \cup \{R_X, \bar{v}R_X\})$, with $C = \{c_1, c_2,\ldots,c_m\}$ and

$R_A = \langle\{\langle x,y\rangle \in A \times A$: there are $i,j \in \{1,\ldots,k\}$ such that
either $i \neq j$, $\langle x,y\rangle \in B_i \times B_j$ and $\langle c_i,c_j\rangle \in R_C$,
or $i = j$ and $\langle x,y\rangle \in R_{B_i}\},A\rangle$.

If $|B_i| \geq 2$ for some $i \in \{1,\ldots,k\}$, then $a \notin B_i$ and $R_{A|_{X-\{b\}}} \in V$ for a $b \in B_i$ by (2.7.1.1). So $R_X = R_{A|_X} \in V$, which is not the case. Hence, $R_A \in \Sigma_1(\{R_C\})$.

By the closedness under restriction of V it follows $R_A \notin V$. Hence, $R_A \in \Sigma_1(\{R_X, \bar{v}R_X\})$ which cannot be the case since $|A| > |X|$.
Hence, $R_A \notin W$.

■

By theorem 2.7.1 it follows that every transitivity condition can be weakened. One might expect that minimal weakenings of the transitivity yield an empty transitivity condition in the limit. That this is not the case is illustrated by the sequence $L(U) \subsetneq_m W(U) \subsetneq_m S_3(U) \subsetneq_m S_4(U) \ldots \subset S(U)$, where the limit situation $S(U)$ is not equal to $V_1(U)$, and hence in the limit situation we do not have an empty transitivity condition.

Let as focus now on the criteria 1 up to 6 imposed on sets of relations which can be classified as sets of orderings.

First we will prove their independence by the following 6 examples.

<u>Example 2.7.2</u> <u>Not closed under permutation</u>

Let $x \in U$ and $W := \{ R_A \in \AA$: there is a $R_B \in L(U)$, such that $R_A := R_B - Id_{\{x\}} \}$.

Obviously W is closed under restriction, conversion, substitution and concatenation. Furthermore, W is non-trivial. Hence, criterion 2,3,4,5 and 6 together do not imply criterion 1.

■

<u>Example 7.3</u> <u>Not closed under conversion</u>

Let $W := \{R_X \in \AA$: R_X is reflexive and R_X is $\langle \bar{a}\ \overline{rs}, \bar{a} \rangle$ - transitive$\}$.

It is straightforward to prove that W is non-trivial, and closed under permutation, concatenation, restriction and substitution.

Let $R_Y := \bar{r}\langle\{\langle x,y\rangle,\langle y,z\rangle,\langle z,y\rangle\},Y\rangle$, where $Y = \{x,y,z\} \in W$.

R_Y is not $\langle \bar{a}\ \overline{rs}, \bar{a} \rangle$ - transitive. R_Y : x.------.z———.y

$\bar{v}R_Y$ is $\langle \bar{a}\ \overline{rs}, \bar{a} \rangle$ - transitive.

So $\bar{v}R_Y \in$, but $R_Y \notin W$.

Hence, W is not closed under conversion.

■

Example 2.7.4 Not closed under restriction

Let $W := \{R_X \in Å :\ R_X \in W(U)$, such that if $Y \subseteq X$ is a maximal indifference class of R_X, then $|Y| \neq 2\}$.

Evidently W is non-trivial and is closed under permutation, conversion, concatenation and substitution.

Furthermore, $L(U) \subset W \subset W(U)$.

Since W(U) is a minimal extension of L(U), W is not close under restriction.

■

Example 2.7.5 Trivial

Å itself is not non-trivial, but Å is closed under permutation, concatenation, restriction, substitution and conversion.

■

Example 2.7.6 Not closed under concatenation

Y_1 is not closed under concatenation.

Y_1 is closed under permutation, restriction, substitution and conversion, and is non-trivial.

■

Example 2.7.7 Not closed under substitution

Let $W = \{R_X \in W(U)$: If Y is a maximal indifference class of R_X, then $|Y| \leq 2\}$.

Obviously W is non-trivial and is closed under permutation, restriction, concatenation and conversion. W is not closed under substitution, since $L(U) \subset W \subset W(U)$ and W(U) is a minimal extension of L(U).

■

By the examples stated above the independence of the criteria 1 up to 6 follows immediately. Furthermore, we notice that the closedness under restriction as well as the closedness under substitution are essential to prove that W(U) is a minimal extension of L(U). Dropping one of these criteria introduces classifiable sets of orderings between L(U) and W(U), as shown in the examples 2.7.4 and 2.7.7. It is left to the reader to prove that the other four criteria are not essential to prove that W(U) is a minimal extension of L(U). However, by example 2.7.6 it becomes clear that dropping the closure under concation introduces sets smaller than L(U).

Let us end this section with a reflection on the criteria introduced in §2.2. The reader should be aware that these indicated properties do not have any absolute truth-value and moreover can probably be formalized in different ways. Hence, we do not argue on the validity of these criteria, whatever that may be. To emphasize this point of view we encourage every reader to find other criteria, which lead perhaps to other classifiable sets of orderings. We end this section with the examination of several slight changes of these criteria.

In this examination we pose two questions:

A. Do there exist weakenings of the criteria 1 up to 6, which lead to a classification system, with acceptable classifiable sets of orderings?

B. Do there exist strengthenings of the criteria 1 up to 6, which lead to another classification system with acceptable sets of orderings?

We start with question A. Notice that a weakening of the criteria will lead to more sets of relations, which can be classified as sets of orderings and that almost all well-known and even less-well-known sets of orderings can be classified as such by the criteria introduced in §2.2. We come to the conclusion that weakening the criteria 1 up to 6 is not of interest to us. Such weakening leads to more fading notions about the phenomenon of ordering. Therefore we will not try to answer question A explicitly.

Now we will try to answer question B. In particular we like to indicate that it is far from easy to introduce strengthenings of the criteria, such that it leads to a satisfactory classification system.

First of all it is noticeable that such a strengthening could be helpful to study the notion of ordering, since according to the classification system introduced in §2.2 there exist many unexpected sets of orderings and several of them are not described by explicit transitivity criteria (e.g. $I_k(U)$).

There will be no discussion of additional new criteria because we could not find other relevant ones. Of course, to overcome the problem about the description of orderings stated above one could demand explicitly that a classifiable set of orderings should satisfy in addition some transitivity conditions. This approach, however, is rejected since because of the variety of these transitivity conditions we finally cannot interpret them.

Hence, we will only discuss strengthenings of the criteria 1 up to 6 themselves. Only strengthenings of criteria 4 and 6 will be discussed since we could not find any relevant strengthenings of criteria 1, 2, 3 and 5.

We start with strengthenings of the closure under concatenation. By criterion 4, $R_A \gg R_B$ is in the classified set of orderings whenever R_A and R_B are in it and A and B are disjoint. To strengthen criterion 4 the disjointness of A and B is weakened. To do this we introduce some notions slightly different from equally named notions in literature.

<u>Definition 2.7.8</u> <u>Better and Maximal elements</u>

Let $R_A \in \mathring{A}$ and $a \in A$

Better $(R_A,a) := \{x \in A : \langle x,a\rangle \in \bar{a}R_A\}$ is the set of <u>better (or preferred) elements to</u> a <u>with respect to</u> R_A.

$\mathrm{Max}(R_A) := \{x \in A : \mathrm{Better}(R_A,x) = \varphi$ and there is a $y \in A$ such that $x \in \mathrm{Better}(R_A,y)\}$ is the set of <u>maximal elements</u> of R_A.

$\mathrm{Min}(R_A) := \mathrm{Max}(\bar{v}R_A)$ is the set of <u>minimal elements</u> of R_A. ∎

In literature $\mathrm{Max}(R_A)$ is defined as the set of elements x such that $\mathrm{Better}(R_A,x)$ is empty. According to our definition $\mathrm{Max}(\varphi_A) = \varphi$, while in literature the set of maximal elements of φ_A is equal to A.

We will now introduce some variations of the concatenation operator:

Definition 2.7.9 Variations of the concatenation operator

Let R_A, R_B ε Å.

2.7.9.1 Let $Max(R_B) = Min(R_A) = C = A \cap B \neq \varphi$ and $R_{A|_C} = R_{B|_C}$.

Then $R_A \gg_1 R_B := <\{<x,y> : <x,y> \varepsilon R_A \vee <x,y> \varepsilon R_B \vee <x,y> \varepsilon (A - C)x(B - C)\}, A \cup B>$.

2.7.9.2 Let $Max(R_B) = Min(R_A) = C = A \cap B$ and $|C| = 1$.

Then $R_A \gg_2 R_B := <\{<x,y> : <x,y> \varepsilon R_A \vee <x,y> \varepsilon R_B \vee <x,y> \varepsilon (A - C)x(B - C)\}, A \cup B>$.

2.7.9.3 Let $Max(R_B) \cap Min(R_A) = C = A \cap B$ and $|C| = 1$.

Then $R_A \gg_3 R_B := <\{<x,y> : <x,y> \varepsilon R_A \vee <x,y> \varepsilon R_B \vee <x,y> \varepsilon (A - C)x(B - C)\}, A \cup B>$.

2.7.9.4 Let $Max(R_B) \cap Min(R_A) = C = A \cap B$ and $|C| = 1$,

$D = Max(R_B) \cup Min(R_A)$, R_D ε Å such that $R_D = \bar{v}R_D$,

$R_{A|_{A \cap D}} = R_{D|_{A \cap D}}$ and $R_{B|_{B \cap D}} = R_{D|_{B \cap D}}$.

Then $\gg_4 (R_A,R_B,R_D) := <\{<x,y> : <x,y> \varepsilon (A - C)x(B - C) \vee <x,y> \varepsilon R_A \vee <x,y> \varepsilon R_B \vee <x,y> \varepsilon R_D\}, A \cup B>$.

■

Although all of these variations at first sight seem to induce reasonable conditions for a set to be classifiable as a set of orderings, they all confront us with odd consequences when doing so. For i ε {1,2,3} we define V $\subseteq$ Å to be $\gg_i$-<u>closed</u> iff $R_A \gg_i R_B$ ε V for all R_A, R_B ε V for which $\gg_i$ can be defined. Furthermore, V is $\gg_4$-<u>closed</u> iff $\gg_4 (R_A, R_B, R_D)$ ε V for all R_A, R_B and R_D for which $\gg_4$ can be defined.

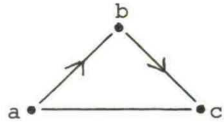
Suppose V is a classified set of orderings. Then the following holds obviously:

if V is $\gg_i$-closed, then V is $\gg_2$-closed for all i ε {3,4,1}. In the following example we will show that several well-known sets of orderings are not $\gg_j$-closed for some j ε {1,2,3,4} and that for all j ε {1,2,3,4} every $\gg_j$-closedness leads to the loss of transtivity properties.

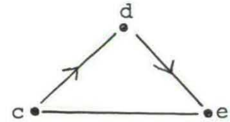
Example 2.7.10

Let $R_A, R_B, R_X, R_Y, R_C, R_D, R_E \in \mathring{A}$ such that the relations have the following graphical representation:

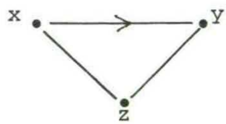
R_A:



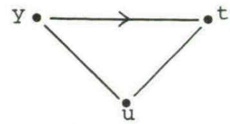
R_B:



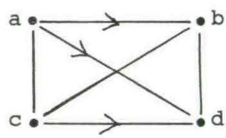
R_X:



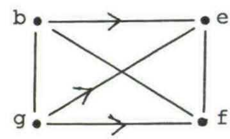
R_Y:



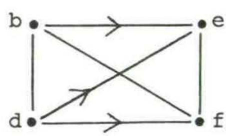
R_C:



R_D:

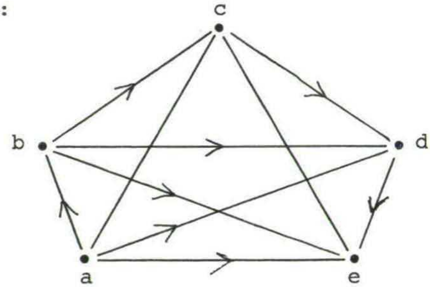


R_E:



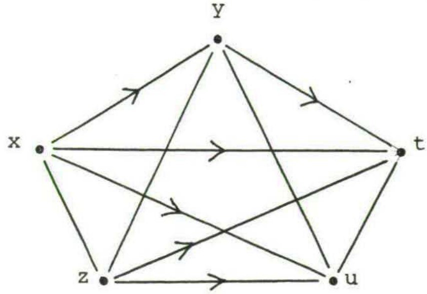
Note that $R_X, R_Y, R_C, R_D, R_E \in S(U)$ and $R_A, R_B \in A_3(U)$. Furthermore, $R_X, R_Y \in S_3(U)$ and $R_C, R_D \in S_4(U)$.

$R_A \gg_2 R_B$:



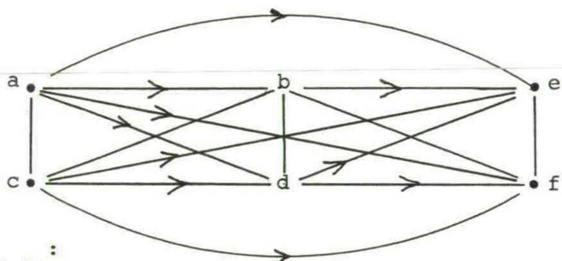
Note that $R_A \gg_2 R_B|_{\{a,c,e\}} \notin A_3(U)$.

$R_X \gg_1 R_Y = R_X \gg_2 R_Y = R_X \gg_3 R_Y = \gg_4(R_X, R_Y, Id_{\{y\}})$:

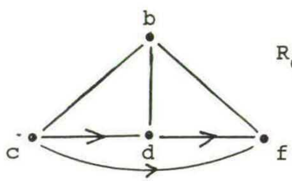


$R_X \gg_2 R_Y \notin S_3(U)$

$R_C \gg_1 R_E$:

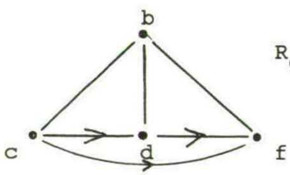


$R_C \gg_1 R_E|_{\{c,b,d,f\}}$:



$R_C \gg_1 R_E|_{\{c,b,d,f\}} \notin S_3(U)$

$R_C \gg_3 R_E|_{\{c,b,d,f\}} = \gg_4(R_C, R_D, Id_{\{b\}})$:



$R_C \gg_3 R_E|_{\{c,b,d,f\}} \notin S(U)$

■

From example 2.7.10 it follows that S(U) is not $\gg_1$-closed, nor $\gg_3$-closed, nor $\gg_4$-closed. Furthermore, from example 2.7.10 it follows that $A_3(U)$ is not $\gg_2$-closed. Since we want to classify at least all well-known sets of orderings, $\gg_1$-closure,

$»_3$-closure and $»_4$-closure cannot be imposed as an criterion on a classifiable set of orderings, otherwise S(U) would not be classifiable anymore. Note that $S_3(U)$ and $S_4(U)$ are not $»_2$-closed. Hence, a $»_2$-closure condition would eliminate some classified sets of orderings. Unfortunately $»_2$-closure implies some odd-results, which we will discuss now.

Observe definition 2.7.9.2. Suppose: R_A, R_B ε Å are such that $»_2$ is well-defined. Then the following holds:

$Max(R_B) = Min(R_A) = \{c\} = A \cap B$ for some c ε U.

Suppose: $\bar{a}R_B$ is transitive. Or stated otherwise, R_B is $\langle\bar{a}\ \bar{a}, \bar{a}\rangle$-classifiable transitive.

Then the following equivalence is easy to prove for all x ε B :

$\langle c,x\rangle \notin \bar{a}R_B$, iff for all y ε B : $\langle x,y\rangle \notin \overline{csca}R_B$.

Hence, if $Max(R_B) = \{c\}$ and R_B is $\langle\bar{a}\ \bar{a}, \bar{a}\rangle$-transitive then $B = B_1 \cup B_2$, such that $B_1 \cap B_2 = \varphi$, $c \in B_1$, $B_1 - \{c\} \neq \varphi$, for all $b_1 \in B_1 - \{c\}$: $\langle c,b_1\rangle \in \bar{a}R_B$, and for all $b_2 \in B_2$ and all $b \in B$: $\langle b_2,b\rangle \in \overline{sca}R_B$. Hence, if R_B is, e.g. a quasi-ordering, and has precisely one maximal element, then the elements which are neither strictly dominated by c (B-Better($\bar{v}R_B$,c)) nor equal to c are disconnected with $\bar{a}R_B$ in $\bar{a}R_B$. Since similar results can be deduced for R_A, it is obvious that $»_2$ is only well-defined for a 'small' subset of Q(A), while » is defined for every 'type' of relation in Å. To illustrate this let R_A, R_B ε Å. Take σ ε S_U such that $\sigma(A) \cap B = \varphi$. Then $\sigma R_A » R_B$ is well-defined.

Hence, $»_2$ is less generally applicable than ». Due to this fact we can observe some odd results. First of all, $T_3(U)$ is $»_2$-closed, since $R_A »_2 R_B$ is well-defined for all R_A, R_B ε $T_3(U)$ iff $Max(R_B) = Min(R_A) = \{c\} = A \cap B$ and $\{c\} > (B - \{c\}) : R_B$ and $(A - \{c\}) > \{c\} : R_A$. Hence, this $»_2$-closure follows immediately from the closure under concatenation of $T_3(U)$. Similarly $C_3(U)$ is $»_2$-closed. But $A_3(U)$ and $S_3(U)$ are not $»_2$-closed. Since $R_A »_2 R_B$ need to be $\langle\overline{rs}^3, \overline{sca}^2\rangle$-transitive or $\langle\overline{rs}^2, \overline{sca}\rangle$-transitive whenever R_A and R_B are $\langle\overline{rs}^3, \overline{sca}^2\rangle$-transitive or respectivily $\langle\overline{rs}^2, \overline{sca}\rangle$-transitive.

This is shown by example 2.7.10.

Secondly it is easy to prove that the only non-$\gg_2$-closed subsets of Q(U), which are classified as sets of orderings and discussed here, are $S_3(U)$ and $S_4(U)$. Hence, $\gg_2$-closure is not very helpful in eliminating several sets of relations, classified as sets of orderings. Furthermore, we observe a strange discontinuity in the non-eliminated sequence:
$S_1(U)$, $S_2(U)$, $S_5(U)$, $S_6(U)$... Hence, $\gg_2$-closure appears to be a rather ad hoc criterion.

Due to these two observations the author decided to incorporate criterion 4 instead of $\gg_2$-closure in the classification mechanism.

Now, having discussed strengthenings of the closure under concatenation, we will discuss strengthenings of the closedness under substitution. First we define two substitution operators.

Definition 2.7.11

Let R_A, $R_B \in \mathring{A}$ and $x \in A$.

2.7.11.1 Suppose $A \cap B = \varphi$ and $Z = A \cup B - \{x\}$.

Then $Sub_1(R_A, x, R_B) := \langle\{\langle a,b\rangle : [\langle a, x\rangle \in R_A \ \&\ a \in B] \vee [\langle a,b\rangle \in R_A \ \&\ a \neq x \ \&\ b \neq x] \vee [\langle x,b\rangle \in R_A \ \&\ b \in B] \vee [\langle a,b\rangle \in R_B]\}, Z\rangle$.

2.7.11.2 Suppose B is a maximal indifference class of R_A and $R_B \in L(U)$.

Then $Sub_2(R_A, B, R_B) := \langle\{\langle a,b\rangle : [\langle a,b\rangle \in R_B] \vee [\langle a,b\rangle \in R_A \ \&\ \langle a, b\rangle \notin (B \times B)]\}, A\rangle$.

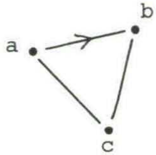
■

In a similar way as in §2.2 we define sub_i-closure for $i \in \{1,2\}$. Now by the following example it is easy to see that S(U) is not sub_1-closed and Q(U) is not sub_2-closed. Hence, as in the introduction to the substitution-operations has been pointed out, allowing non-reversible relations as substitution arguments leads to violations of various transitivity properties. Because of this fact S(U) and Q(U) are not sub_1-closed and not sub_2-closed respectivily. Hence, in the classification system these operations are not admissible.

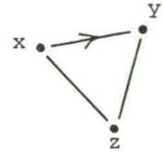
Example 2.7.12

Let R_A, R_B, R_C, R_D ε $\mathring{A}$ such that

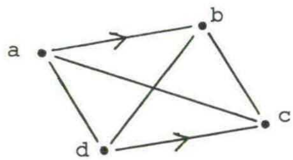
R_A:



R_B:

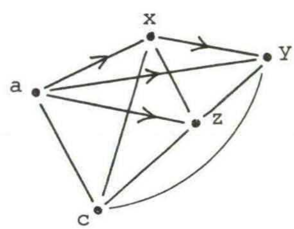


R_C:

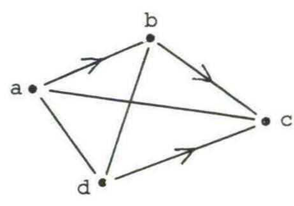


R_D:

Then $Sub_1(R_A,\ b,\ R_B)$:



and $Sub_2(R_C,\ \{b,c\},\ R_D)$:



Hence, $Sub_1(R_A,\ b,\ R_B) \notin S(U)$ and $Sub_2(R_C,\ \{b,c\},\ R_D) \notin Q(U)$.

■

A more subtle substitution operation than the one introduced in 2.2.19, in which the substitutes are either a total indifference class or a total incomparability class, is left out of the discussion here. As mentioned before, with this more subtle substitution mechanism the theorems stated in this chapter remain almost the same, but their proofs become much more complicated. (See earlier versions of this monograph).

Summarizing we conclude that by the classification system introduced in §2.2 all well-known sets of orderings can be classified. Unfortunately, several (in fact infinitely many) 'new' sets of relations can be classified as sets of orderings. At first sight all attempts to reduce this number of new sets, which can be classified as sets of orderings, failed. To our intuition this is due to the fact that the attempts affect transitivity properties of orderings. Furthermore, a systematic analysis of the monadic operations based on local information has been worked out. This analysis made possible a formulation of a general transitivity condition. Also there has been explored a characterization of minimal extensions of a given set of orderings and along with this extension a minimal weakening of the transitivity condition of these orderings. So we have developed a (preliminary) classification system, which gives us insight in the phenomenon of ordering. This insight is of vital importance in the following chapters, but to the author's opinion it is not yet satisfactorily explored, since we have the following open question:

Is the set of criteria 1 up to 6 replaceable by another set 1' up to n', such that all the new criteria can be interpreted very easily in a meaningful manner and such that classifying sets of relations according to these new criteria leads to the following results:

(1) All well-known sets of orderings can be classified as a set of orderings according to the new criteria, and

(2) The number of sets which can be classified is much smaller than the number of sets classified above.

CHAPTER 3 DISCRETE METRIC SPACES

§ 3.1 Metric spaces

In the previous chapter an investigation into the notion of ordering is made to establish the important concepts of the model of constitutional decision rules: individual preferences and preferences of society. In chapter 4 the found classification system gives rise to a framework, in which new impossibility theorems can be derived and almost all known impossibility theorems involving social welfare functions are incorporated. These welfare functions have the independence of irrelevant alternatives property.

In this chapter continuity properties for welfare functions are introduced. Several of these continuity properties turn out to be weaker conditions for welfare functions than the independence of irrelevant alternatives condition. In the next chapter impossibilities for continuous welfare functions are deduced also.

Continuity is often related to functions with a domain and a range which are not discrete. However, here the domain and codomain of the studied functions are even finite, hence discrete. For this reason a more careful introduction of the continuity conditions is established here.

Continuity is a topological property: A function f is continuous (with respect to the topologies τ_{domain} and $\tau_{codomain}$), iff for every open set O (according to $\tau_{codomain}$) it holds that the set of originals, whose images under f are in O, is open (according to τ_{domain}). Clearly the continuity property depends strongly on the topologies τ_{domain} and $\tau_{codomain}$. In this chapter new topologies are generated by means of distance functions. Let V be a set of elements and d a distance function on V. Then a well known topology is based on the set of balls. A ball with centre $v \in V$ and radius α (where α is a positive real number) is a set $B(v,\alpha) := \{x \in V: d(x,v) < \alpha\}$. If V is finite this leads to a topology, where every singleton subset of V is open. Consequently it follows that every subset of V is open.

So, if the domain and codomain are finite metric spaces and τ_{domain} and $\tau_{codomain}$ are topologies based on the set of balls, then every function from this domain to that codomain is continuous. Continuity introduced in this way does not in any way impose constraints on a function (for the finite case of course).

To make use of metric spaces in connection with meaningful continuity properties, the topologies based on these distance functions are deduced more subtly. The information of points which have, according to the metric space, the smallest possible distance, will also determine whether or not a set is open. In this way non-standard topologies are obtained. Of course we have to pay a price for this: the Hausdorff property does not hold in general for these new topologies. In this chapter distance functions are extended to the information as mentioned above. Then the new topologies are defined by means of these extended distance functions. Furthermore, a special type of metric spaces, named full metric spaces, are studied and it is shown how to fill a metric space. Finally, continuity with respect to these new topologies is characterized and a preparation of the impossibilities for continuous constitutional decision rules is developed. This preparation concerns properties of metric spaces. This is why a whole chapter is devoted to metric spaces.

In this section some basic notions are introduced.

<u>Definition 3.1.1</u> <u>Distance function</u>

Let V be an arbitrary set and R the set of real numbers.

A <u>distance function</u> d on V is a function from V x V to the set R, such that for all $x,y,z \in V$:

3.1.1.1 $d(x,y) \geq 0$ (no negative distances),

3.1.1.2 $d(x,y) = 0$, iff $x = y$ (equal elements have a zero distance),

3.1.1.3 $d(x,y) = d(y,x)$ (symmetry),

3.1.1.4 $d(x,z) + d(z,y) \geq d(x,y)$ (triangle inequality).

■

Since this standard definition is given in many handbooks about topology or analysis no further comment is given here.

In connection with distance functions the following notions appear useful.

Definition 3.1.2

Let V be an arbitrary set and d a distance function on V.

3.1.2.1. For arbitrary $x \in V$ and $u \in R$, $u > 0$, the ball with centre x and radius u (in V with respect to d) is equal to $\{a \in V : d(x,a) < u\}$.

Notation: $B(x,u,V,d)$, or whenever V and d are known $B(x,u)$.

3.1.2.2. The diameter of V with respect to d is equal to $\sup \{d(x,y) : x,y \in V\}$.

Notation: $diam(V,d)$.

3.1.2.3. The meshwidth of V with respect to d is equal to $\inf \{d(x,y) : x,y \in V, x \neq y\}$.

Notation: $mesh(V,d)$.

■

The ball with centre x and radius u is also called u-neighbourhood of x. This notion as well as the notion diameter are standard notions. The meshwidth is perhaps not generally used, because in the frequently used metric spaces it is zero. If V is finite, then $mesh(V,d) = \min\{d(x,y) : x,y \in V, x \neq y\} > 0$. $mesh(V,d)$ is the smallest positive distance between two elements of V. The meshwidth is illustrated by the following example.

Example 3.1.3 Meshwidth, Diameter, Neighbourhood

Let the Euclidian distance, d, on R^n, the n-fold cartesian product of R, be defined as follows:

$d: R^n \times R^n \to R$, such that for all $\langle x_1,x_2,\ldots,x_n\rangle,\langle y_1,y_2,\ldots,y_n\rangle \in R^n$:

$$d(\langle x_1,x_2,\ldots,x_n\rangle,\langle y_1,y_2,\ldots,y_n\rangle) := \\ = \sqrt{(x_1-y_1)^2+(x_2-y_2)^2+\ldots+(x_n-y_n)^2}$$

Let Z be the set of integers, i.e., $Z = \{..-2,-1,0,1,2,3..\}$. It is easy to prove that $mesh(Z \times Z, d) = 1$,

$$B(\langle 0,0\rangle,2,Z \times Z,d) = \{\langle 0,0\rangle,\langle -1,0\rangle,\langle 0,-1\rangle,\langle 1,0\rangle,\langle 0,1\rangle, \\ \langle -1,1\rangle,\langle 1,1\rangle,\langle 1,-1\rangle,\langle -1,-1\rangle\},$$

and $diam(Z \times Z,d) \notin R$.

The diameter is not in R, since Z x Z is not bounded. If the diameter is not in R it is infinite: notation ∞. Hence, $diam(Z \times Z,d) = \infty$. If $V = \{x \in R : x \geq 0 \;\&\; x \leq k\}$, where k is a fixed positive number in Z, then mesh(V,d) = 0 and diam(V,d) = k.

If V = {x ε R : there is a n ε N or Z\{0}, such that $x = n - (^1/_n)$}, then mesh(V,d) = 1. But d(x,y) > 1 for all x,y ε V, with $x \neq y$.

■

In the previous example the following well known obvious fact is used: the restriction of a distance function is a distance function.

To be complete let us recall the notion of the metric space. A <u>metric space</u> M is an ordered pair <V,d>, where d is a distance function on the set V. Notation: M = <V,d>.

§ 3.2 Topologies based on distance functions

In this section information deduced from a distance function is used to construct non-standard topologies. This information is described by graphs and is essentially richer than the information about distance functions which is standardly used. The standard information is whether an element y is or is not in a u-neighbourhood of another element x. In finite metric spaces $\langle V,d \rangle$ ($|V| \in Z$) this information is not subtle enough to construct meaningful topologies. As stated before, since this information leads to the topology based on the set of balls, the topology generated by this information states that every subset of V is open. Therefore, it is trivial. On the other hand, the topologies constructed here have not the Hausdorff property.

The extra information which is used to build these topologies is described by a betweenness-relation based on a distance function d. An element z is between the elements x and y, iff $d(x,z) + d(z,y) = d(x,y)$ and $z \notin \{x,y\}$. This betweenness information is incorporated in the set on which a topology is defined. For instance, if V is finite and $x \in V$, then a basic open surrounding of x is the set of x together with the information about the elements $y \in V$, which are as near as possible to x (i.e. elements y, with $d(x,y) = \mathrm{mesh}(V,d)$). This information can be formalized as a set of edges in a graph. So, a basic open set is a vertex together with several edges ending in that vertex.

To make these rough notions clear let some formalization be started. First the notions graph and topology are recalled. Since these are standard mathematical attributes no special attention is paid to them.

Definition 3.2.1 Topology

Let V be a set.

A topology on V is a collection τ of subsets of V, such that:

3.2.1.1 $V, \varphi \in \tau$,

3.2.1.2 every union of elements of τ is in τ,

i.e., $\cup \{X : X \in \beta\} \in \tau$ for all $\beta \subseteq \tau$, and

3.2.1.3 the intersection of finitely many elements of τ is in τ,

i.e., $\cap \{X : X \in \beta\} \in \tau$ for all $\beta \subseteq \tau$, with $|\beta| \in Z$.

The pair $<V,\tau>$ is a topological space, an element X of τ is called open and an element $Y \in 2^V$ is closed, iff $(V - Y)$ is open.

■

Definition 3.2.2 Graph

A (simple) graph G is an ordered pair $<V,E>$, such that:

$E \subseteq \{\{x,y\} \in 2^V : x \neq y\}$.

V is the set of vertices and E is the set of edges. If $\{x,y\} \in E$, then x and y are the end-points of $\{x,y\}$.

■

The following notion is used to link distance functions with graphs.

Definition 3.2.3 Between

Let d be a distance function on V and $x,y \in V$. Then an element z of V is between x and y (with respect to d), iff $d(x,z) + d(z,y) = d(x,y)$ and $z \notin \{x,y\}$.

■

An element z is between x and y, iff z is not equal to x or y and there is a shortest path from x to y via z. For such elements z the triangle inequality (3.1.1.4) is an equality.

Example 3.2.4

Let d be the Euclidian distance on R x R.

The circle C around $<0,0>$, with radius 1, equals $\{<x,y> \in R \times R : d(<x,y>,<0,0>) = 1\}$.

For all $x,y \in C$ and for all $z \in C$, z is not between x and y.

Take $I = \{<x,y> \in R \times R : y = 0 \;\&\; 0 \leq x \leq 1\}$.

Then $<\frac{1}{2},0>$ is between $<0,0>$ and $<1,0>$.

■

This betweenness relation is a notion frequently used in geometry. In an Euclidean space the points between two other points determine the open line segment bounded by those two points. Since it is such a natural relation no further explanation is spent to it.

This betweenness relation gives rise to the neighbouring information used in the topologies introduced hereafter. The first step in utilizing this betweenness information is to formalize this knowledge about the metric space in a graph. The second step incorporates this formal information in the given distance function of the metric space. In other words, the distance function is extended to the formal betweenness information. Then, according to this extended distance function, a topology is defined.

<u>Definition 3.2.5</u> <u>Neighbourhood graph</u>

Let $M = \langle V,d \rangle$ be a metric space.

The <u>neighbourhood graph</u> G_M <u>of</u> M is the graph $\langle V,E_M \rangle$, where

$E_M := \{\{x,y\} \subseteq V : x \neq y$ and there is not an element $z \in V$, such that z is between x and y$\}$.

■

If $\{x,y\} \in E_M$, then there is no shortest path from x to y via any other element z in $\langle V,d \rangle$. So x and y are true neighbours of each other.

<u>Example 3.2.6</u>

Let d be the Euclidean distance on R x R and C as in example 3.2.4. Then the neighbourhood graph G_M of $M = \langle C,d \rangle$ equals $\langle C,\{\{x,y\} : x,y \in C \ \&\ x \neq y\}\rangle$.

The neighbourhood graph $G_{M'}$ of $M' = \langle Z,d' \rangle$ equals $\langle Z,\{\{n,n+1\} : n \in Z\}\rangle$, where $d'(x,y) = |x - y|$ for all $x,y \in R$.

Furthermore, the neighbourhood graph $G_{M''}$ of $M'' = \langle R,d' \rangle$ equals $\langle R,\varphi \rangle$. $G_{M''}$ is totally unconnected.

■

It is obvious that for every metric space $M = \langle V,d \rangle$ we have: if $x,y \in V$, $x \neq y$ and $d(x,y) = mesh(V,d)$, then $\{x,y\} \in E_M$. So

$\{x,y\}$ is an edge of G_M, if the distance of x and y equals the mesh. Note that the converse does not hold. For instance, take $\langle 1,0\rangle$ and $\langle -1,0\rangle$ in C and $M = \langle C,d\rangle$. Then $\{x,y\} \in E_M$, but $\mathrm{diam}(C,d) = d(\langle 1,0\rangle,\langle -1,0\rangle) > \mathrm{mesh}(C,d)$.

Now a distance function is defined between the edges of a graph, whose vertices are already endowed with a distance function.

<u>Definition 3.2.7</u> <u>Extended distance function</u>

Let $M = \langle V,d\rangle$ be a metric space and let $G = \langle V,E\rangle$ be a graph. The <u>extended distance function</u> d_G <u>of</u> d is a function from $(V \cup E) \times (V \cup E)$ defined as follows:

$$d_G(\{x,y\},\{a,b\}) := \begin{cases} 0 \text{ if } \{x,y\} = \{a,b\} \\ \frac{1}{4}\,[d(x,a) + d(x,b) + d(y,a) + d(y,b)] \\ \text{otherwise.} \end{cases}$$

For convenience an element $v \in V$ is identified with $\{v,v\}$ here.

■

Although d_G is obviously a well-defined function, it is still open whether or not d_G is a distance function, and if so whether or not it is an extension of d. In the following proposition these questions are answered. The proof of the following proposition follows easily from the fact that d is a distance function, therefore it is left to the reader.

<u>Proposition 3.2.8</u> d_G <u>is an extension of</u> d

Let $M = \langle V,d\rangle$ be a metric space, let $G = \langle V,E\rangle$ be a graph and let d_G be the extended distance function of d. Then for all x,y and z in $V \cup E$:

3.2.8.1 $d_G(x,y) \geq 0$,

3.2.8.2 $d_G(x,y) = 0$, iff $x = y$,

3.2.8.3 $d_G(x,y) = d_G(y,x)$,

3.2.8.4 $d_G(x,y) + d_G(y,z) \geq d_G(x,z)$, and

3.2.8.5 $d_{G|_V} = d$.

Hence, d_G is a distance function on $V \cup E$ and it is an extension of d.

■

Before introducing a base for a topology some examples will illustrate d_G.

Example 3.2.9

Let $x,y \in C$, $m,n \in Z$ and let d and d'' be as in example 3.2.6.

$$d_{G_{<C,d>}}(\{x,x\},\{x,y\}) = \tfrac{1}{4}[d(x,x) + d(x,y) + d(x,x) + d(x,y)]$$
$$= \tfrac{1}{2}d(x,y).$$

$$d_{G_{<C,d>}}(\{x,y\},\{x,z\}) = \tfrac{1}{4}[d(x,z) + d(x,y) + d(y,z)].$$

$$d_{G_{<Z,d'>}}(\{n,n\},\{n,n+1\}) = \tfrac{1}{2}.$$

$$d_{G_{<Z,d'>}}(\{n,n\},\{m,m+1\}) = \tfrac{1}{4}[|n-m| + |n-(m+1)| + |n-m| + |n-(m+1)|]$$
$$= \tfrac{1}{2}|n-m| + \tfrac{1}{2}|n-m-1|.$$

$$d_{G_{<Z,d'>}}(\{n,n+1\},\{n,n-1\}) = \tfrac{1}{4}[|n-n+1| + |n+1-n| + |n+1-n+1|].$$

■

Next an assumption, which is often used later on, is stated.

Assumption 3.2.10

Suppose $M = <V,d>$ is a metric space and mesh = mesh(V,d). Furthermore let $G_M = <V,E_M>$ be the neighbourhood graph of M, and let d_{G_M} be the extended distance function of d.

■

The following type of balls is used to define a base of the topology, in which we are interested.

Definition 3.2.11 Mesh-edged-balls

Assume 3.2.10.

$W \subseteq V \cup E_M$ is a mesh-edged-ball, iff there is an $\{x,y\} \in W$ and $u > 0$, such that:

3.2.10.1 $W = B(\{x,y\}, u, V \cup E_M, d_{G_M})$, and

3.2.10.2 for all $\{a,a\} \in V \cup E_M$:
if $u > \tfrac{1}{2}[d(x,a) + d(a,y)]$,
then $u > \tfrac{1}{2}[d(x,a) + d(a,y) + \text{mesh}]$.

■

$W \subseteq V \cup E_M$ is a mesh-edged-ball, iff it is a ball with centre $\{x,y\}$ and radius u, where u has property (3.2.10.2).
If $diam(V,d) < \infty$, then $B(\{x,y\}, u, V \cup E_M, d_{G_M})$ is a mesh-edged-ball for all $\{x,y\} \in V \cup E_M$ and $u > diam(V, d) + \frac{1}{2}mesh$. In fact this ball is equal to $V \cup E_M$. The prefix 'mesh-edged' suggests that the ball is bordered by mesh-edges (i.e., edges with length equal to the meshwidth). The next proposition illustrates that this suggestion is correct. In this proposition it is stated that if a vertex of G_M is in a mesh-edged-ball, then all its adjacent edges of length meshwidth are also in that ball.

<u>Proposition 3.2.12</u>

Assume 3.2.10.

Let $W = B(\{x,y\}, u, V \cup E_M, d_{G_M})$ be a mesh-edged-ball and $\{a,a\} \in W$. Then $\{a,b\} \in W$ for all $b \in V$, with $d(a,b) = mesh$.

<u>Proof of proposition 3.2.12</u>

Let all the variables be as above. Suppose $\{a,a\} \in W$, $b \in V$ and $d(a,b) = mesh$. Then we have to prove that $d_{G_M}(\{a,b\},\{x,y\}) < u$. Now by the triangle inequality and the symmetry property of distance functions it follows:

$$\begin{aligned} d_{G_M}(\{a,b\},\{x,y\}) &= \tfrac{1}{4}[d(a,x) + d(a,y) + d(b,y) + d(b,x)] \\ &\le \tfrac{1}{4}[2d(a,x) + 2d(a,y) + 2d(a,b)] \\ &= \tfrac{1}{2}[d(a,x) + d(a,y) + mesh]. \end{aligned}$$

Since $\{a,a\} \in W$ it follows that

$u > d_{G_M}(\{a,a\},\{x,y\}) = \frac{1}{2}[d(a,x) + d(a,y)]$.

Now by the mesh-edgedness of W, it follows that

$u > d_{G_M}(\{a,b\},\{x,y\})$.

■

Note that if $Mesh(V,d) = 0$, then every ball is a mesh-edged-ball. Let $mesh(V,d) := mesh > 0$ and $a \in V$.
If $B(\{a,a\}, u, V \cup E_M, d_{G_M}) := W$ is a mesh-edged-ball, then $\{a,a\} \in W$. Since $u > \frac{1}{2}[d(a,a) + d(a,a)] = 0$ it then follows that $u > \frac{1}{2}[d(a,a) + d(a,a) + mesh] = \frac{1}{2}mesh$. On the other hand, it is

obvious that $B(\{a,a\}, \text{mesh}, V \cup E_M, d_{G_M})$ is a mesh-edged-ball around $\{a,a\}$. This ball is called the <u>mesh-ball</u> around $\{a,a\}$. Let $\{x,y\} \in E_M$. Then $B(\{x,y\}, \frac{1}{2}\text{mesh}, V \cup E_M, d_{G_M}) = \{\{x,y\}\}$ is also a mesh-edged-ball. It is also called the <u>mesh-ball</u> around $\{x,y\}$. The collection of these mesh-balls forms a subbase for the following topology, whenever mesh > 0.

<u>Definition 3.2.13</u> <u>d-induced topology</u>

Assume 3.2.10.

Then the <u>d-inducedtopology</u> on $E_M \cup V$, τ_d is the collection of subsets, X, of $E_M \cup V$, such that X is the union of a collection of mesh-edged-balls.

■

Before discussing this topology, it is necessary to prove that τ_d is indeed a topology.

<u>Proposition 3.2.14</u>

Assume 3.2.10 and furthermore, let τ_d be the d-induced topology on $V \cup E_M$.

Then τ_d is a topology on $V \cup E_M$.

<u>Proof of proposition 3.2.14</u>

Two cases are distinguished.

<u>Case 1</u> mesh(V, d) = 0.

In this case every ball is mesh-edged and it is well known that τ_d is a topology.

<u>Case 2</u> mesh(V, d) =: mesh > 0.

It is sufficient to prove that (3.2.1.1) up to (3.2.1.3) hold for τ_d.

(3.2.1.1) $\varphi \in \tau_d$ by the assumption that the union over an empty collection yields the empty set.

$V \cup E_M \in \tau_d$ since $V \cup E_M$ is equal to the union of mesh-balls around the elements of $V \cup E_M$.

(3.2.1.2) obviously holds.

(3.2.1.3) It is sufficient to prove that the intersection of two mesh-balls is a mesh-ball or empty. This however is evident.

■

The topology τ_d depends on the distance function d and its induced betweenness-relation. For instance, if V is finite, then a metric space $M = \langle V,d \rangle$ will correspond with a topology τ_d, which is different from the standard topology based on the set of balls. On the other hand, if M is nowhere discrete (i.e., for all distinct $x,y \in V$, there is a $z \in V$ such that z is between x and y), then τ_d is equal to the topology based on the set of balls. Let these facts be illustrated by the following example.

<u>Example 3.2.15</u>

Let $M = \langle V,d \rangle$ a metric space. The topology $\tau_{M \text{ standard}}$ on V is defined as follows:

X is in $\tau_{M \text{ standard}}$, iff X is the union of a collection of balls.

Take $M^1 = \langle Z, d_1 \rangle$, $M^2 = \langle R \times R, d_2 \rangle$ and $M^3 = \langle R \times R, d_3 \rangle$, where

$d_1(z_1,z_2) := |z_1 - z_2|$ for all $z_1, z_2 \in Z$,

$d_2(\langle x,y \rangle,\langle a,b \rangle) := \sqrt{((x-a)^2 + (y-b)^2}$ for all $\langle a,b \rangle,\langle x,y \rangle \in R \times R$,

$d_4(x,a) := |\{x,a\}| - 1$ for all $x,a \in R$,

$d_5(b,y) := \dfrac{|b - y|}{|b - y| + 1}$ for all $b,y \in R$,

$d_6(x,a) := |x - a|$ for all $x,a \in R$, and

$d_3(\langle x,y \rangle,\langle a,b \rangle) := d_4(x,a) + d_5(b,y) + d_6(x,a)$ for all $\langle a,b \rangle,\langle x,y \rangle \in R \times R$.

It is straightforward to prove that d_1, d_2, d_4, d_5 and d_6 are distance functions. Furthermore, since d_4, d_5 and d_6 are distance functions it follows that d_3 is also a distance function.

Obviously, $\tau_{M^1 \text{standard}} = 2^Z$ and $\tau_{M^2 \text{standard}}$ is the topology, which we are all familiar with. $\tau_{M^2 \text{standard}}$ is based on the daily-life geometrical experience of the plane: the points in a circular disk are closer to the centre of that disk, than the points outside that disk. If we think of the plane to be arranged otherwise and base the topology on this other arrangement, then less familiar topologies are obtained, even in the case of standard topology approach.

For instance, let the points of the plane R x R, be arranged according to a lexicographical ordering, which results in the following property: a point $\langle x,y\rangle$ is closer to a point $\langle x,z\rangle$ on the same vertical line through $\langle x,y\rangle$ than to any other point not on that line.

As the reader might have guessed, d_3 establishes this arrangement. $d_3(\langle x,y\rangle,\langle a,b\rangle) \geq d_4(x,a)$ and $d_4(x,a) = 1$, if and only if $x \neq a$.

On the other hand
$$\begin{aligned} d_3(\langle x,y\rangle,\langle x,z\rangle) &= d_4(x,x)+d_5(y,z)+d_6(x,x) \\ &= d_5(y,z) \\ &= \frac{|y - z|}{|y - z| + 1} < 1. \end{aligned}$$

This illustrates that d_3 has the above mentioned property. Note that $B(\langle x,y\rangle, 1, R \times R, d_3) = \{\langle x,b\rangle \in R \times R: b \in R\}$.

Hence, $B(\langle x,y\rangle, 1, R \times R, d_3)$ is open in $\tau_{M^3 standard}$, but $B(\langle x,y\rangle, 1, R \times R, d_3)$ is closed in $\tau_{M^2 standard}$.

Obviously $G_M2 = G_M3 = \langle R \times R, \varphi\rangle$ and therefore $\tau_{d_3} = \tau_{M^3 standard}$ and $\tau_{d_2} = \tau_{M^2 standard}$.

The τ_d topologies do not always give rise to new topologies. Now consider M^1, then $E_M1 = \{\{n,n+1\} : n \in Z\}$. Hence,

$\tau_{d_1} = \{W \subseteq Z \cup E_M1 : \text{if } \{n,n\} \in W, \text{ then } \{n,n+1\},\{n,n-1\} \in W\}$.

Clearly $\tau_{d_1} \neq \tau_{M^1 standard}$. Notice that every $W \in \tau_{M^1 standard}$ is closed in τ_{d_1}.

■

This section is concluded by some statements about τ_d topologies. These statements are discussed in the following remarks.

<u>Remark 3.2.16</u> <u>Separability</u>

Assume 3.2.10 and furthermore, let τ_d be the topology on $V \cup E_M$ based on d.

τ_d is weakly separable (i.e., for every twó elements in $V \cup E_M$ it holds: there is as open set in τ_d containing precisely one of these elements). Hence, τ_d is a so called T_0-topology (see e.g., Kelley [1955] and Császár [1978]). The proof of this follows immediately from the facts that:

(1) if mesh(V,d) = 0, then it is standard.

(2) if mesh(V,d) > 0, then the mesh-ball around an edge is equal to the singleton of that edge and the mesh-ball around a vertex does not contain any other vertex.

In general τ_d is not separable (Separable means that for every pair of distinct elements a,b in V U E_M, there are open sets A, B in τ_d, such that a ε A, b $\not\varepsilon$ A, a $\not\varepsilon$ B and b ε B). Take for instance M^1 = <Z, d_1> (as in the previous example). Then {1,1} and {1,2} are not separable, since every open surrounding in τ_{d_1} of {1,1} contains {1,2}.

However, the following equivalence holds:

τ_d is separable, iff d(x,y) > mesh(V,d) for all x,y ε V, with x $\neq$ y.

The proof of this assertion is as follows.

It is obvious that:

τ_d is separable, iff for all {x,y} ε E_M, there is a u ε R, with ½mesh(V,d) < u < ½d(x,y).

Hence, τ_d is separable, iff d(x,y) > mesh(V,d) for all x,y ε V, with x $\neq$ y.

In general τ_d is not strongly separable (i.e., for distinct elements a,b in V U E_M there are open sets A, B in τ_d, such that a ε A, b ε B and A ∩ B = φ).

Take M^1 (see example 3.2.15). τ_{d_1} is not even separable.

However the following equivalence holds:

τ_d is strongly separable, hence a <u>Hausdorff-space</u>, iff τ_d is separable.

The proof of this equivalence is similar to that of the foregoing one.

τ_d is a weakly separable space and it is a Hausdorff-space, iff mesh(V,d) is a real infimum (i.e., it is not a minimum).

Edges of length meshwidth can not topologically be separated in τ_d from their endpoints. Or intuitively stated, the information of a minimal distance can not be separated from either of the points, which have this distance. This inseparability induces 'strong' properties on continuous functions.

■

Remark 3.2.17 When is τ_d equal to $\tau_{M\ standard}$?

In this remark this question will be answered.

Assume 3.2.10.

Since $E_M \in \tau_d$ and $E_M \notin \tau_{M\ standard}$, if $E_M \neq \varphi$, it holds:
if $\tau_d = \tau_{M\ standard}$, then $E_M = \varphi$.

Now the converse is proved (i.e., if $E_M = \varphi$ then $\tau_d = \tau_{M\ standard}$).

Two cases are distinguished.

Case 1 mesh(V,d) = 0.

Then obviously $\tau_d = \tau_{M\ standard}$, by definition of both topologies.

Case 2 mesh(V,d) = mesh > 0.

It is proved that this can not be the case.

Take $x,y \in V$, with $x \neq y$. Then $d(x,y) \leq k \cdot \text{mesh}$ for some $k \in Z$.

Since $E_M = \varphi$, there is a z between x and y.

Hence, $d(x,z) < (k-1) \cdot \text{mesh}$ or $d(y,z) < (k-1) \cdot \text{mesh}$.

By induction it follows evidently that there are $a,b \in V$, such that $0 < d(a,b) < \text{mesh}$.

This contradicts the definition of mesh(V,d).

Therefore, $\tau_d = \tau_{M\ standard}$ iff $E_M = \varphi$.

In the last part of this remark it is proved that $\tau_{M\ standard}$ is in the induced topology of τ_d. This is clearly established by the following equivalence:

$X \in \tau_{M\ standard}$ iff for all $x \in X$ there is a mesh-edged-ball $W_x \subseteq V \cup E_M$ and $\cup\{W_x : x \in X\} \cap V = X$.

Proof of the equivalence:

(Only if) Suppose $X \in \tau_{M\ standard}$.

Then for all $x \in X$ there is a real number $u_x > 0$, such that $B(x,a,V,d) \subseteq X$.

$$\text{Let } w_x := \begin{cases} \text{mesh}(V,d) & \text{iff mesh}(V,d) > 0 \\ u_x & \text{iff mesh}(V,d) = 0 \end{cases}$$

Take $W_x := B(\{x,x\}, w_x, V \cup E_M, d_{G_M})$

Obviously W_x is a mesh-edged-ball for every $x \in X$.

It is sufficient to prove that $(\cup\{W_x : x \in X\}) \cap V = X$.

Since $x \in W_x$ and $x \in V$, for all $x \in X$ it follows that $X \subseteq (\cup\{W_x : x \in X\}) \cap V$.

Suppose $y \in V$ and $y \in \cup \{W_x : x \in X\}$.

Then $y \in V$ and for some $x \in X$, $y \in W_x$.

Hence, $\mathrm{mesh}(V,d) = 0$ and evidently $y \in X$, or $\mathrm{mesh}(V,d) > 0$, $x = y$, and $y \in X$.

So $\cup\{W_x : x \in X\}) \cap V \subseteq X$.

Thus $\cup\{W_x : x \in X\}) \cap V = X$.

(if) Suppose $X = (\cup\{W_x : x \in X\}) \cap V$, where W_x is a mesh-edged-ball in $V \cup E_M$ for all $x \in X$.

Without loss of generality suppose

$W_x = B(\{x,x\}, w_x, V \cup E_M, d_{G_M})$.

Take $W'_x := B(x, w_x, V, d)$.

Since $W'_x = W_x \cap V$ it is obvious that

$X = \cup\{W'_x : x \in X\} \in \tau_M$ standard.

■

§ 3.3 Full metric spaces.

Although the subject of full metric spaces could have been treated, along with the results and definitions about metric spaces in § 3.1, it is developed here in order to acquaint the reader more carefully with this property. A metric space is full if for every distance t in the range of the distance function and segment between x and y, with length greater or equal to t, there is a z on that segment such that the distance between x and z is equal to t.

This is more or less equivalent to a construction principle in geometry, which makes it possible to transfer a distance of a segment by a pair of compasses to another segment. Because of this resemblance the terms used to introduce the fulness property have a geometrical nature.

The content of this section is as follows: first the notion of segment along with some of its properties is discussed. By virtue of this notion the fulness property can be defined. After this definition some characterizations of full metric spaces are studied. Finally, a procedure is developed to construct a full metric space from special metric spaces.

The discrete full metric spaces, enriched with topologies as discussed in § 3.2, lead to continuity properties, which have a simple character. This is shown in § 3.4. Since these continuity properties play an important rôle in the following chapter, they are studied extensively in this chapter.

Let us start with the introduction of the fulness property. As noted before the notion of segment has to be developed first.

<u>Definition 3.3.1</u> <u>Segment</u>

Let $M = \langle V,d\rangle$ be a metric space and $a,b \in V$.
The <u>segment</u> $[a,b]_M := \{z \in V : d(a,z) + d(z,b) = d(a,b))\}$ is the set of elements in V, which are in {a,b} or between a and b.

■

Let d be the Euclidean distance on R (see 3.1.3) and $M = \langle R, d\rangle$. Then $[0,1]_M$ is the closed interval [0,1] of real numbers between 0 and 1. Let $M^3 = \langle R \times R, d^3\rangle$ as defined in example 3.2.15. Then it is straightforward to calculate

$[\langle 0,0\rangle,\langle 1,0\rangle]_{M^3} = \{\langle 0,0\rangle,\langle 1,0\rangle\}$.

This demonstrates that although intervals are related to segments, they are not the same. Let the following proposition bring more light on the notion of segment.

<u>Proposition 3.3.2</u> <u>Properties of segments</u>.

Let $M = \langle V,d\rangle$ be a metric space and $x,y \in V$. Then the following holds:

3.3.2.1 $x,y \in [x,y]_M$,

3.3.2.2 $[x,y]_M = [y,x]_M$,

3.3.2.3 if $z \in [x,y]_M$, then $[x,z]_M \subseteq [x,y]_M$, and

3.3.2.4 if $[x,y]_M = [x,z]_M$, then $z = y$.

<u>Proof of proposition 3.3.2</u>

(3.3.2.1) and (3.3.2.2) are trivial.

(3.3.2.3) Let $z \in [x,y]_M$ and $t \in [x,z]_M$.

It is sufficient to prove: $d(x,y) \geq d(x,t) + d(t,y)$.

Since $z \in [x,y]_M$ it follows that:

$d(x,z) + d(z,y) = d(x,y)$.

Since $t \in [x,z]_M$ it follows that:

$d(x,t) + d(t,z) = d(x,z)$.

Hence, $d(x,t) + d(t,z) + d(z,y) = d(x,y)$,

so $d(x,t) + d(t,y) \leq d(x,y)$.

(3.3.2.4) Let $[x,y]_M = [x,z]_M$.

We have to prove that $z = y$.

Since $z \in [x,y]_M$ and $y \in [x,z]_M$ we have

$d(x,z) + d(z,y) = d(x,y)$ and $d(x,y) + d(y,z) = d(x,z)$.

Hence, $2 \cdot d(y,z) = 0$, and $y = z$.

■

In general the following does not hold:

if $x,y \in [a,b]_M$, then $[x,y]_M \subseteq [a,b]_M$, and

if $[a,b]_M = [x,y]_M$, then $\{a,b\} = \{x,y\}$.

The reader is invited to construct counter examples.

Before stating the fulness condition a notational convention is introduced. Let R be a relation from V to W and $V' \subseteq V$, then

$R(V') := \{w \in W :$ there is a $v' \in V'$, such that $\langle v',w\rangle \in R\}$ is the image of V' under R. Often this notation is used when R is a function. For instance, if $M = \langle V,d\rangle$ is a metric space, then $d(V \times V)$ is the set of all achievable distances on V.

Now the notion full metric space can be defined.

Definition 3.3.3 Full metric space

Let $M = \langle V,d\rangle$ be a metric space.

Then M is full, iff for all $t \in d(V,V)$, and all $x,y \in V$, with $d(x,y) \geq t$, there is a $z \in [x,y]_M$, such that $d(x,z) = t$.

■

The metric space $M = \langle V,d\rangle$ is full, iff for distances $t \in d(V \times V)$ and segments $[x,y]_M$, with $t \leq d(x,y)$, there is an element z in $[x,y]_M$ at distance t from x. Although the reader might curiously look for full metric spaces and spaces without this property, he is asked to postpone this activity, since the following theorems simplify this investigation.

Since our interest is focussed on discrete metric spaces, we first characterize discrete full metric spaces.

Theorem 3.3.4 Character of discrete full metric spaces.

Assume 3.2.10, with mesh > 0.

Then (3.3.4.1), (3.3.4.2) and (3.3.4.3) are equivalent, where

3.3.4.1 M is full,

3.3.4.2 for all $x,y \in V$: if $d(x,y) >$ mesh, then $[x,y]_M \neq \{x,y\}$,

3.3.4.3 for all $x,y \in V$, there is a $l \in Z$, such that $d(\{x\} \times [x,y]_M) = \{k \cdot \text{mesh} : k \in Z \;\&\; 0 \leq k \leq l\}$.

Proof of theorem 3.3.4

(3.3.4.1) → (3.3.4.2) This implication follows evidently by definition 3.3.1 and 3.3.3.

(3.3.4.2) → (3.3.4.3) It is sufficient to prove (3.3.4.4). This will be done by induction on l.

3.3.4.4 For all $x,y \in V$ and all $l \in Z$, with $l \geq 0$ and $d(x,y) \leq l \cdot \text{mesh}$, there is a $k \in Z$, with $0 \leq k \leq l$, such that $d(x,y) = k \cdot \text{mesh}$.

Proof of (3.3.4.4)

- $l = 0$ This is a trivial case.

- Suppose the induction hypothesis holds for l.

Let $l \cdot mesh < d(a,b) \leq (l+1) \cdot mesh$.

It is sufficient to prove that $d(a,b) = (l+1) \cdot mesh$.

There are two cases

Case 1 $l = 0$.

Then by the assumption that $mesh > 0$ and the definition of meshwidth it follows that $d(a,b) = mesh$.

Case 2 $l > 0$.

By (3.3.4.2) there is a $z \in [a,b]_M$, such that $z \notin \{a,b\}$.

Hence, $d(a,z) + d(z,b) = d(a,b)$, $d(a,z) > 0$ and $d(z,b) > 0$.

Since $d(a,b) \leq (l+1) \cdot mesh$ it follows that $d(a,z) \leq l \cdot mesh$ or $d(z,b) \leq l \cdot mesh$. Suppose without loss of generality that:

$0 < d(a,z) \leq l \cdot mesh$.

By the induction hypothesis it follows that $0 < d(a,z) = k_1 \cdot mesh$ and $k_1 \leq l$.

Hence, $d(b,z) \leq l \cdot mesh$.

Again by the induction hypothesis it follows that

$0 < d(z,b) = k_2 \cdot mesh$ and $k_2 \leq l$.

$$\begin{aligned} \text{Hence, } l \cdot mesh < d(a,z) + d(z,b) &= (k_1 + k_2) \cdot mesh \\ &= d(a,b) \leq (l + 1) \cdot mesh. \end{aligned}$$

Hence, $d(a,b) = (l + 1) \cdot mesh$.

(3.3.4.3) -> (3.3.4.1) This implication is trivial.

■

In general we have the following characterization:

Theorem 3.3.5 Character of full metric spaces

Let $M = \langle V,d \rangle$ be a metric space.

Then M is full, iff for all $t_1,t_2 \in d(VxV)$, and all $x,y \in V$:

if $|t_1 - t_2| \leq d(x,y)$, then $|t_1 - t_2| \in d(\{x\} \times [x,y]_M)$.

Proof of theorem 3.3.5

(Only if) Suppose M is full, $t_1,t_2 \in d(VxV)$, $x,y \in V$ and $|t_1 - t_2| \leq d(x,y)$.

Then we have to prove that there is a $z \in [x,y]_M$, such that $|t_1 - t_2| = d(x,z)$.

Since M is full, it is sufficient to prove that there are $a,b \in V$, such that $d(a,b) = |t_1 - t_2|$.

Without loss of generality suppose that $t_1 \geq t_2$.
Since $t_1, t_2 \in d(VxV)$ there are $a_1, b_1, a_2, b_2 \in V$, such that $d(a_1,b_1) = t_1$ and $d(a_2,b_2) = t_2$.
Hence, there is an element c in $[a_1,b_1]_M$, such that $d(c,b_1) = t_2$.
Hence, $d(a_1,c) = t_1 - t_2 = |t_1 - t_2|$ and we are done.

(if) This implication is simple to prove.

■

From (3.3.5) as an immediate consequence we obtain:

Theorem 3.3.6

Let $M = \langle V,d\rangle$ be a full metric space.
Then for all $t_1, t_2 \in d(VxV)$: $|t_1 - t_2| \in d(VxV)$.

■

Theorem 3.3.4 shows that discrete full metric spaces have a simple nature. Furthermore, by theorem 3.3.5 one can check whether or not a metric space is full. The reader may check this fulness property for the metric spaces introduced before. Since the real topic in this chapter is in the field of discrete spaces no further attention is paid to the character of full metric spaces with meshwidth zero.

The next step in this section is to construct a full metric space from an arbitrary one. This cannot be done in a simple way. When the arbitrary chosen metric space does not have some metric properties, it may be difficult to construct a full metric space embedding the first one. The following example clarifies the troubles indicated above.

Example 3.3.7

Let $M = \langle Z \times Z, d\rangle$ be the metric space, where d is the Euclidean distance function.
Clearly M is not full, since $1 = \text{mesh}(Z \times Z, d)$ and $d(\langle 0,0\rangle,\langle 1,1\rangle) = \sqrt{2} \notin \{1 \cdot k : k \in \{0,1,2....\}\}$.
Let $M' = \langle V',d'\rangle$ be a full metric space, such that $(Z \times Z) \subseteq V'$ and $d'_{|Z \times Z} = d$.
Is M' desirable as an extension of M, which is full?
Suppose $\text{mesh}(V',d') =: \text{mesh} > 0$. By theorem 3.3.4 there are $k_1, k_2 \in \{0,1,2....\}$, such that $1 = k_1 \cdot \text{mesh}$ and $\sqrt{2} = k_2 \cdot \text{mesh}$.

Hence, $\sqrt{2} = k_2/k_1 \in Q$. But this contradicts our knowledge about numbers. Hence, mesh(V',d') = 0.

So $d'(V'xV') \neq d((ZxZ)x(ZxZ))$ and in M' the fulness condition is stronger than in M.

Furthermore, $\tau_{M'standard} = \tau_{d'}$ is a Hausdorff topology, where $\tau_d \neq \tau_{Mstandard}$ and τ_d is not a Hausdorff space.

■

For reasons of simplicity the problem indicated above is avoided. The construction of a full metric space is only completed in those cases where the image of the distance function of the full space is equal to that of the space to be filled. To be more precise: the filling construction is only applied on a metric space with the following property.

Definition 3.3.8 Full image

Let M = <V,d> be a metric space. Furthermore, let M_E := <d(VxV), d_E> be a metric space, where d_E is the Euclidean distance on R.

M has a full image, iff M_E is full.

■

The following theorem characterizes full image spaces.

Theorem 3.3.9 Character of full image spaces

Let M = <V,d> be a metric space and M_E = <d(VxV),d_E> the metric space with Euclidean distance function d_E.

Then (3.3.9.1), (3.3.9.2) and (3.3.9.3) are equivalent, where

3.3.9.1 M has a full image,

3.3.9.2 $d_E(d(VxV)xd(VxV)) = d(VxV)$, and

3.3.9.3 for all $t_1,t_2 \in d(VxV)$: $|t_1 - t_2| \in d(VxV)$.

Proof of theorem 3.3.9

(3.3.9.1) -> (3.3.9.2) Suppose M has a full image.

Then M_E is full.

Since $0 \in d(VxV)$ it holds for every $t \in d(VxV)$, that $t = |t - 0| = d_E(t,0) \in d_E(WxW)$, where W = d(VxV).

Hence, $d(VxV) \subseteq d_E(WxW)$.

Suppose $t_1,t_2 \in W$, such that $t_1 \geq t_2$.
It is to prove that $d_E(t_1,t_2) = |t_1 - t_2| \in W$.
Now $t_1,t_2 \in d(VxV) \subseteq d_E(WxW)$ and $|t_1 - t_2| \leq d_E(0,t_1)$.
Hence, by theorems 3.3.4 and the fulness of M_E it follows that $|t_1 - t_2| \in d_E(\{0\} \times [0,t_1]_{M_E})$.
Evidently (by our notion about numbers) we have

$$|t_1 - t_2| \in [0,t_1]_{M_E} \subseteq d(VxV) = W.$$

(3.3.9.2) → (3.3.9.3) This implication is simple to prove.
(3.3.9.3) → (3.3.9.1) Suppose for all $t_1,t_2 \in d(VxV)$:
$|t_1 - t_2| \in d(VxV)$, let $t_3,t_4 \in d_E(VxV)$ and $x,y \in W$, such that $|t_3 - t_4| \leq d_E(x,y)$ and $x \leq y$, $t_3 \leq t_4$.
We have to prove that $|t_3 - t_4| \in d_E(\{x\}x[x,y]_{M_E})$.
Hence it is sufficient to prove that $x + |t_3 - t_4| \in d(VxV)$.
Since $t_3,t_4 \in d_E(WxW)$, there are t_3^1,t_3^2,t_4^1 and $t_4^2 \in W = d(VxV)$, such that $t_3 = d_E(t_3^1,t_3^2) = |t_3^1 - t_3^2|$ and $t_4 = d_E(t_4^1,t_4^2) = |t_4^1 - t_4^2|$.
Hence, by our assumption $t_3,t_4 \in d(VxV) = W$.
Note that

$$\begin{aligned} x + |t_3 - t_4| &= y - [(y - x) - (t_3 - t_4)] \\ &= |y - ||y - x| - |t_3 - t_4|||. \end{aligned}$$

But then, again using the assumption, it follows that $x + |t_3 - t_4| \in d(VxV)$.

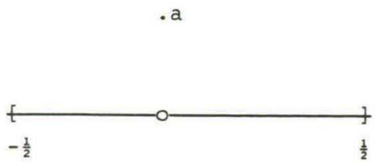
■

In the following example it is shown that the topological properties of the space may change in another way, when filling it.

<u>Example 3.3.10</u>

Let $M = \langle V,d\rangle$ be a metric space, such that $V = \{a\} \cup W$, where $a \notin W$ and $W = \{x \in Q : x \neq 0 \;\&\; -\frac{1}{2} \leq x \leq \frac{1}{2}\}$ and d is defined as follows:
$d(a,a) = 0$
$d(x,a) = d(a,x) = 1 + x^2$ for all $x \in W$.
$d(x,y) = |x - y|$ for all $x,y \in W$.
Clearly d is a distance function.

M can be pictured as follows:



M is not full, since there is no point on the segment $[-\frac{1}{4},\frac{1}{4}]$ at a distance $\frac{1}{4}$ from $\frac{1}{4}$ on the segment, where such a point is on segment $[\frac{1}{4},\frac{1}{2}]$.
To fill M we have at least to plug in a point zero in W.
Consider the segment $[a,x]_M$ for $x \in W$.
If $y \in [a,x]_M$ and $y \notin \{a,x\}$, then $d(a,y) + (y,x) = d(a,x)$.
Hence, $1 + y^2 + |y - x| = 1 + x^2$. So $|y - x| = x^2 - y^2$.
But then $x + y = 1$ or $x + y = -1$. This is impossible.
Hence, $[a,x]_M = \{a,x\}$ for all $x \in W$.
Therefore, if we would define the distance between a and zero, say $d'(a,0)$, equal to 1, then by the same arguments as above we would introduce a totally new segment between a and 0. This of course may not happen, since this would change the segment structure of the metric space.
On the other hand it is not difficult to prove that $d'(a,0)$ should be equal to 1. This will be shown now, hence we have a contradiction.
For all $n \in \{2,3,....\}$ we have $d'(a,0) + d(0,1/_n) \geq d(a,1/_n)$ and $d'(a,0) \leq d(0,1/_n) + d(a,1/_n)$.
Hence, for all $n \in \{2,3,....\}$:
$d'(a,0) \geq 1 + 1/_{n^2} - 1/_n$ and $d'(a,0) \leq 1 + 1/_{n^2} + 1/_n$.
Hence, $d'(a,0) = 1$.

■

As indicated in the previous example we may ruin the structure of a metric space by filling it. This can occur because of creating intersections of disjunct segments or by introducing new ones. To be able to be more precise about the structure of a metric space a new notion is introduced.

Definition 3.3.11 Simple segments

Let $M = \langle V,d \rangle$ be a metric space, $a,b \in V$ and $[a,b]_M$ a segment of M.

$[a,b]_M$ is a simple segment, iff for all $c \in [a,b]_M$:

$[a,b]_M \subseteq [a,c]_M \cup [c,b]_M$.

■

A segment $[a,b]_M$ is simple, iff for all its elements it holds that they subdivide this segment in two segments which cover $[a,b]_M$. These segments play an important rôle when filling a metric space. If they characterize the structure of a metric space, then it is possible to fill this space such that the filled space has the same structure as the original one. Before any further insight in this approach is revealed, it is necessary to develop more insight in the notion of simple segment. The following theorem gives this insight.

Theorem 3.3.12 Properties of simple segments

Let $M = \langle V,d \rangle$ be a metric space, let $a,b,c \in V$ and let $c \in [a,b]_M$.

3.3.12.1 $[a,b]_M \subseteq [a,c]_M \cup [c,b]_M$, iff

$[a,c]_M = \{x \in [a,b]_M : d(a,x) \le d(a,c)\}$ and

$[c,b]_M = \{x \in [a,b]_M : d(a,x) \ge d(a,c)\}$.

3.3.12.2 $[a,b]_M$ is a simple segment, iff for all $c \in [a,b]_M$:

$[a,c]_M = \{x \in [a,b]_M : d(a,x) \le d(a,c)\}$ and

$[c,b]_M = \{x \in [a,b]_M : d(a,x) \ge d(a,c)\}$.

3.3.12.3 If $[a,b]_M$ is a simple segment and $c,e \in [a,b]_M$, such that $d(a,c) \le d(a,e)$, then

$$[c,e]_M = \{x \in [a,b]_M : d(a,c) \le d(a,x) \le d(a,d)\}.$$

3.3.12.4 If $a,b,c,e \in V$, $[a,b]_M$ and $[c,e]_M$ are simple segment and $x,y \in [a,b]_M \cap [c,e]_M$, then segment $[x,y]_M$ is simple and $[x,y]_M \subseteq [a,b]_M \cap [c,e]_M$.

3.3.12.5 If $[a,b]_M$ is a simple segment, then for all $c \in [a,b]_M$:

$$\{c\} = \{z \in [a,b]_M : d(a,z) = d(a,c)\}.$$

Proof of theorem 3.3.12

3.3.12.1 (only if) Suppose $[a,b]_M \subseteq [a,c]_M \cup [c,b]_M$.

The $[a,c]_M \cup [c,b]_M = [a,b]_M$.

Let $x \in [a,c]_M$.

Then $d(a,x) + d(x,c) = d(a,c)$ and $[a,c]_M \subseteq [a,b]_M$.

Since $d(x,c) \geq 0$, it follows that $d(a,x) \leq d(a,c)$.

So $[a,c]_M \subseteq \{x \in [a,b]_M: d(a,x) \leq d(a,c)\}$.

Let $x \in [a,b]_M$, with $d(a,x) \leq d(a,c)$.

Then $x \in [a,c]_M$ or $x \in [c,b]_M$.

Suppose $x \in [c,b]_M$.

Then $d(x,c) + d(x,b) = d(c,b)$.

Since $d(a,x) \leq d(a,c)$ we have

$$\begin{aligned} d(x,b) &= d(a,b) - d(x,a) \\ &\geq d(a,b) - d(a,c) \\ &= d(b,c) \\ &= d(x,b) + d(x,c). \end{aligned}$$

So $x = c$.

Then $\{x \in [a,b]_M: d(a,x) \leq d(a,c)\} \subseteq [a,c]_M$.

Hence, $[a,c]_M = \{x \in [a,b]_M: d(a,x) \leq d(a,c)\}$.

Similarly it follows that

$[b,c]_M = \{x \in [a,b]_M: d(a,x) \geq d(a,c)\}$.

This completes the proof of this implication.

(if) This implication is simple to prove.

(3.3.12.2) is simple to prove.

(3.3.12.3) Let $[a,b]_M$ be a simple segment and $e \in [a,b]_M$.

First it is to prove that $[a,e]_M$ is a simple segment.

Let $x \in [a,e]_M$.

It is sufficient to prove that

$[a,x]_M = \{z \in [a,e]_M : d(a,z) \leq d(a,x)\}$ and

$[e,x]_M = \{z \in [a,e]_M : d(a,z) \geq d(a,x)\}$.

Note that $x \in [a,e]_M \subseteq [a,b]_M$.

Since $[a,b]_M$ is simple, it follows that

$$\begin{aligned} [a,x]_M &= \{z \in [a,b]_M : d(a,z) \leq d(a,x)\} \\ &= \{z \in [a,b]_M : d(a,z) \leq d(a,x) \text{ and } d(a,z) \leq d(a,e)\} \\ &= \{z \in [a,b]_M : d(a,z) \leq d(a,x) \text{ and } z \in [a,e]_M\} \\ &= \{z \in [a,e]_M : d(a,z) \leq d(a,x)\}. \end{aligned}$$

Suppose $z \in [x,e]_M$.

We have to prove that $d(a,z) \geq d(a,x)$.

Now $d(a,x) + d(x,e) = d(a,e)$,

$d(a,z) + d(z,e) = d(a,e)$ and

$d(x,z) + d(z,e) = d(x,e)$.

Hence, $d(a,x) + d(x,z) + d(z,e) = d(a,z) + d(z,e)$.

Since $d(x,z) \geq 0$ we obtain $d(a,x) \leq d(a,z)$.

Hence, $[x,e]_M \subseteq \{ z \in [a,e]_M : d(a,z) \geq d(a,x)\}$.

Suppose $d(a,z) \geq d(a,x)$ and $z \in [a,e]_M$.

We have to prove that $d(e,z) + d(z,x) = d(e,x)$.

Now $z \in [a,b]_M$, which is simple.

Therefore $[a,z]_M = \{t \in [a,b]_M : d(a,z) \geq d(a,t)\}$.

So $x \in [a,z]_M$ and $d(a,x) + d(x,z) = d(a,z)$.

$$
\begin{aligned}
\text{Now } d(a,e) &\leq d(a,x) + d(x,e) \\
&\leq d(a,x) + d(x,z) + d(z,e) \\
&= d(a,z) + d(z,e) \\
&= d(a,e).
\end{aligned}
$$

But then $d(x,z) + d(z,e) = d(x,e)$ and

$\{z \in [a,e]_M : d(a,z) \geq d(a,x)\} \subseteq [x,e]_M$.

It is proved that $[a,e]_M$ is simple.

Similarly we have $[a,c]_M$ is simple.

Hence, $c \in [a,e]_M$ and $[c,e]_M$ is simple.

Furthermore, it follows that

$$
\begin{aligned}
[c,e]_M &= \{x \in [a,e]_M : d(a,x) \geq d(a,c)\} \\
&= \{x \in [a,b]_M : d(a,x) \geq d(a,c) \text{ and } d(a,x) \leq d(a,e)\} \\
&= \{x \in [a,b]_M : d(a,c) \leq d(a,x) \leq d(a,e)\}.
\end{aligned}
$$

(3.3.12.4) is simple to prove.

(3.3.12.5) Suppose $[a,b]_M$ is simple and let $c \in [a,b]_M$.

Suppose $d(a,z) = d(a,c)$.

Then by (3.3.12.2), we have $z \in [a,c]_M$.

Hence, $d(a,z) + d(z,c) = d(a,c) = d(a,z)$.

So $z = c$ and consequently $\{c\}$ equals

$\{z \in [a,b]_M : d(a,z) = d(a,c)\}$.

■

Now we are able to develop the construction mechanism, which fills metric spaces. First of all we will define the range of this mechanism, that is what kind of properties the full space should have, whenever it is constructed by the mechanism. These properties guarantee that the metric structure of the old space as well as the topological one is preserved. Such a full space will be called a regular full extension. We will prove that there is at most one regular full extension for each metric space. Hence, if the mechanism is applicable, then it yields a unique result.

Having this result we will indicate a domain for the mechanism and describe it. Finally we will prove that the proposed construction yields a regular full metric space for any metric space in the indicated domain.

Definition 3.3.13 Regular full extension

Let $M = \langle V,d \rangle$ and $M' = \langle V',d' \rangle$ be two metric spaces.

M' is a regular full extension of M, iff (3.3.13.1), (3.3.13.2), (3.3.13.3) and (3.3.13.4) hold, where

3.3.13.1 there is an injective function h from V to V', such that $d(x,y) = d'(h(x),h(y))$ for all $x,y \in V$, and M' is full,

3.3.13.2 for all $x \in V'$ there are $a,b \in h(V)$, such that $x \in [a,b]_M$ and $[a,b]_M$ is simple,

3.3.13.3 for all $a,b \in V$: $[a,b]_M$ is simple, iff $[h(a),h(b)]_{M'}$ is simple, and

3.3.13.4 for all $x,y \in V'$ and all $a,b \in h(V)$, with $[a,b]_{M'}$ is simple, $x \in [a,b]_{M'}$ and $y \notin [a,b]_{M'}$, it holds that $h(V) \cap [a,b]_{M'} \cap [x,y]_{M'} \neq \varphi$.

∎

(3.3.13.1) preserves the structure of M in M'. Hence, by (3.3.13.1) we may speak of an extension of M. By (3.3.13.2) it follows that a point $x \in V' - V$ is not added to V unnecessarily. Hence, each point in $x \in V' - V$ is essential. By (3.3.13.3) and (3.3.13.4) M' has the same structural properties as M, that is no new simple segments are introduced in M'.

Theorem 3.3.14 Uniqueness of regular full extensions

Let $M = \langle V,d \rangle$, $M' = \langle V',d' \rangle$, and $M'' = \langle V'',d'' \rangle$ be metric spaces.

If M' and M" are both regular full extensions of M, then there is a bijective function g from V' to V", such that for all $x,y \in V'$: $d'(x,y) = d''(g(x),g(y))$.

∎

The uniqueness of a regular full extension is to be interpreted as uniqueness modulo isomorphism: if M' and M" are both regular full extensions of M, then M' and M" have the same metric properties.

<u>Proof of theorem 3.3.14</u>

Suppose M' and M" are both regular full extension of M.

Let h' be the injection from V to V' and h" the injection from V to V" according to (3.3.13.1) up to (3.3.13.4).

First we discuss two steps.

<u>Step 1</u> Let $a,b,c,e \in V$, $x \in V'$ and $y \in V''$, such that

$x \in [h'(a),h'(b)]_{M'}$, which is simple,

$y \in [h''(a),h''(b)]_{M''}$, which is simple,

$x \in [h'(c),h'(e)]_{M'}$, which is simple, and

$d'(x,h'(a)) = d''(y,h''(a))$.

Then $y \in [h''(c),h''(e)]_{M''}$ and $d''(h''(c),y) = d'(h'(c),x)$.

<u>Proof of step 1</u>

Suppose a,b,c,e,x and y satisfy the above condition.

It is proved that $y \in [h''(c),h''(e)]_{M''}$ and

$$d''(h''(c),y) = d'(h'(c),x).$$

Since $x \in [h'(a),h'(b)]_{M'}$ and $y \in [h''(a),h''(b)]_{M''}$

it follows that

$d'(h'(a),x) + d'(h'(b),x) = d'(h'(a),h'(b)) = d(a,b)$

$d''(h''(a),y) + d''(h''(b),y) = d''(h''(a),h''(b)) = d(a,b)$

Hence, $d'(h'(b),x) = d''(h''(b),y)$.

<u>Claim 1.1</u> There are $p,q \in V$, such that

$x \in [h'(p),h'(q)]_{M'} \subseteq [h'(a),h'(b)]_{M'} \cap [h'(c),h'(e)]_{M'}$

<u>Proof of claim 1.1</u> Take p = c, if $h'(c) \in [h'(a),h'(b)]_{M'}$;

otherwise by (3.3.13.4) there is a $p \in V$, such that

$h'(p) \in [h'(a),h'(b)]_{M'} \cap [x,h'(c)]_{M'}$.

Hence, $[x,h'(p)]_{M'} \subseteq [h'(a),h'(b)]_{M'} \cap [x,h'(c)]_{M'}$.

Take q = e, if $h'(e) \in [h'(a),h'(b)]_{M'}$; otherwise by (3.3.13.4) there is a $q \in V$, such that

$h'(q) \in [h'(a),h'(b)]_{M'} \cap [x,h'(e)]_{M'}$.

Hence, $[x,h'(q)]_{M'} \subseteq [h'(a),h'(b)]_{M'} \cap [x,h'(e)]_{M'}$.

By theorem 3.3.12 it follows that

$x \in [h'(p),h'(q)]_{M'} \subseteq [h'(a),h'(b)]_{M'} \cap [h'(c),h'(e)]_{M'}$.

This proves claim 1.1.

Hence, such $p,q \in V$ exist.

Furthermore, by (3.3.12) $[h'(p),h'(q)]_{M'}$ is simple.

Now the following two assertions hold by (3.3.12):

(1) $h'(p) \in [h'(a),x]_{M'}$ or $h'(p) \in [h'(b),x]_{M'}$, and

(2) $h'(p) \in [h'(c),x]_{M'}$ or $h'(p) \in [h'(e),x]_{M'}$.

Without loss of generality suppose $h'(p) \in [h'(a),x]_{M'}$ and $h'(p) \in [h'(c),x]_{M'}$.

Hence, $d'(h'(c),x) = d'(h'(c),h'(p)) + d'(h'(p),x)$ and

$$d'(h'(a),x) = d'(h'(a),h'(p)) + d'(h'(p),x).$$

Hence,

$$\begin{aligned} d'(h'(a),h'(p)) &\le d'(h'(a),x), \\ d(a,p) &\le d'(h'(a),x), \\ d''(h''(a),h''(p)) &\le d''(h''(a),y). \end{aligned}$$

Since $p \in [a,b]_M$ it follows that $h''(p) \in [h''(a),h''(b)]_M$ and by (3.3.12), it holds that $h''(p) \in [h''(a),y]_{M''}$.

Hence,

$$\begin{aligned} d''(h''(a),y) &\le d''(h''(a),h''(p)) + d''(h''(p),y), \\ d'(h'(a),y) &\le d(a,p) + d''(h''(p),y), \\ d'(h'(a),y) &\le d'(h'(a),h'(p)) + d''(h''(p),y). \end{aligned}$$

Hence, $d''(h''(p),y) = d'(h'(p),x)$.

Since $d'(h'(b),x) = d''(h''(b),y)$ and obviously $h'(q) \in [x,h'(b)]_{M'} \cap [x,h'(e)]_{M'}$, it follows similarly that $d''(h'(q),y) = d'(h'(q),x)$.

Hence, $y \in [h''(p),h''(q)]_{M''}$.

Since $q,p \in [c,e]_{M'}$ which is simple by (3.3.13.3) and the fact that $[h'(c),h'(e)]_{M'}$ is simple, it follows by (3.3.12) that $[p,q]_M \subseteq [c,e]_M$.

Hence, $y \in [h''(p),h''(q)]_{M''} \subseteq [h''(c),h''(e)]_{M''}$.

So $y \in [h''(c),h''(e)]_{M''}$.

This proves one part of step 1.

The following holds:

$$d'(h'(c),h'(p)) \le d'(h'(c),x) \le d'(h'(c),h'(q)) \le d'(h'(c),h'(e)).$$

So:

$$d''(h''(c),h''(p)) \le d''(h''(c),y) \le d''(h''(c),h''(q)) \le d''(h''(c),h''(e)).$$

Since $[h'(c),h'(e)]_{M'}$ is simple, it follows by (3.3.13.3) that $[h''(c),h''(e)]_{M''}$ is simple.

Therefore since $h'(p) \in [h'(c),h'(e)]_{M'}$, by (3.3.13.1) it follows that $h''(p) \in [h''(c),h''(e)]_{M''}$ and by (3.3.12) it follows that $h''(p) \in [h''(c),y]_{M''}$.

Hence, $d''(h''(c),y) = d''(h''(c),h''(p)) + d''(y,h''(p))$
$= d'(h'(c),h'(p)) + d'(x,h'(p))$
$= d'(h'(c),x)$.

This proves step 1.

Step 2 Let $a,b,c,e \in V$, $x \in V'$ and $y \in V''$, such that
$x \in [h'(a),h'(b)]_{M'}$, which is simple,
$y \in [h''(a),h''(b)]_{M''}$, which is simple,
$y \in [h''(c),h''(e)]_{M''}$, which is simple, and
$d'(x,h'(a)) = d''(y,h''(a))$.
Then $x \in [h'(c),h'(e)]_{M'}$ and $d''(h''(c),y) = d'(h'(c),x)$.

Proof of Step 2 This proof is similar to the proof of step 1.

Now g will be defined.

Let $x \in V'$. Then there are $a,b \in V$, such that $x \in [h'(a),h'(b)]_{M'}$, which is simple.

Hence, $[h''(a),h''(b)]_{M''}$ is simple and since M'' is full, there is a $y \in [h''(a),h''(b)]_{M''}$, with $d'(h'(a),x) = d''(h''(a),y)$.

Take $g(x) = y$.

g is well defined by step 1.

g is injective by step 2.

g is surjective by the fulness of M' and M''.

Hence, g is a bijection from V' to V''.

Take $x,y \in V'$.

For reasons of symmetry it is sufficient to prove that:
$d'(x,y) \geq d''(g(x),g(y))$.

There are $a,b,c,e \in V$, such that $x \in [h'(a),h'(b)]_{M'}$ and $y \in [h'(c),h'(e)]_{M'}$ and both segments are simple.

Hence, $g(x) \in [h''(a),h''(b)]_{M''}$, which is simple,
$d'(x,h'(a)) = d''(g(x),h''(a))$,
$g(y) \in [h''(c),h''(e)]_{M''}$, which is simple, and
$d''(y,h''(c)) = d'(x,h'(c))$.

There are two cases:

Case 1 $y \in [h'(a),h'(b)]_{M'}$

Then obviously $g(y) \in [h''(a),h''(b)]_{M''}$ and
$d'(x,y) = |d'(h'(a),x) - d'(h'(a),y)| =$
$= |d''(h''(a),g(x)) - d''(h''(a),g(y))| =$
$= d''(g(x),g(y))$.

Case 2 $y \notin [h'(a),h'(b)]_{M'}$

Then there is a $p \in V$ by (3.3.13.4), such that

$h'(p) \in [h'(a),h'(b)]_{M'} \cap [x,y]_{M'}$.

Without loss of generality suppose $h'(p) \in [x,h'(b)]_{M'}$.

Now there are again two subcases:

Case 2^A $h'(p) \in [h'(c),h'(e)]_{M'}$

Without loss of generality suppose $h'(p) \in [h'(c),y]_{M'}$.

Then $d'(x,h'(p)) + d'(h'(p),y) = d'(x,y)$ and

$d''(g(x),h''(p)) + d''(h''(p),g(y)) \geq d''(g(x),g(y))$

Case 2^B $h'(p) \notin [h'(c),h'(e)]_{M'}$

Then there is a $q \in V$, such that

$h'(q) \in [h'(c),h'(e)]_{M'} \cap [h'(p),y]_{M'}$.

But then we have:

$$\begin{aligned} d'(x,y) &= d'(x,h'(p) + d'(h'(p),y) \\ &= d''(g(x),h''(p)) + d'(h'(p),h'(q)) + d'(h'(q),y) \\ &= d''(g(x),h''(p)) + d''(h''(p),h''(q)) + d''(h''(q),g(y)) \\ &\geq d''(g(x),g(y)). \end{aligned}$$

This completes the proof.

■

Before giving the construction mechanism we will indicate a domain of this mechanism. Possibly it may be taken somewhat larger but this is to the author's opinion only a marginal extension of the domain chosen here.

Definition 3.3.15 Weakly full

Let $M = \langle V,d \rangle$ be a metric space

M is weakly full, iff (3.3.15.1) and (3.3.15.2) hold, where

3.3.15.1 for all $x,y \in M$ and all $t \in d(VxV)$, with $t \leq d(x,y)$, there are $p,q \in [x,y]_M$, such that $d(p,x) \leq t \leq d(q,x)$ and $[p,q]_M = \{p,q\}$

3.3.15.2 for all $t_1,t_2 \in d(VxV)$: $|t_1 - t_2| \in d(VxV)$ (M has a full image).

■

First of all let the name be explained. If a metric space M is weakly full, then by (3.3.15.1) we can locate this not yet full parts as discrete simple segments. Of course M is then almost full hence weakly filled.

By example (3.3.7) it is shown that (3.3.15.2) is essential and by (3.3.10) it is shown that (3.3.15.1) is essential. The essentiallity of (3.3.15.2) is clear by theorem 3.3.9, that of (3.3.15.1) follows because there are not such p and q in V for $x = -\frac{1}{4}$, $y = \frac{1}{4}$ and $t = \frac{1}{4}$ in the metric space of example (3.3.10).

Finally we are able to give the construction mechanism.

Theorem 3.3.16 Constructable regular full extension

Let $M = \langle V,d \rangle$ be a weakly full metric space.

Then there is a regular full extension $M' = \langle V',d' \rangle$ of M.

Proof of theorem 3.3.16

Let $W = \{\langle a,b,t \rangle \in V \times V \times d(V \times V) :$

$[a,b]_M$ is simple and $d(a,b) \geq t\}$.

$\langle a,b,t \rangle$ can be interpreted as the point on $[a,b]_M$ at distance t from a. It is clear that W would fill V, but there can be more notations for the same point, e.g., $\langle a,a,0 \rangle$ and $\langle b,a,d(a,b) \rangle$. Therefore, an equivalence relation is defined, such that an equivalence class coincides with one point in V' and the different notations for that point are in the same equivalence class.

Let $\langle a,b,t \rangle \in W$. By the weakly fulness of M there are $p,q \in [a,b]_M$, such that $[p,q]_M = \{p,q\}$ and $d(a,p) \leq t \leq d(a,q)$. Since $[a,b]_M$ is simple by assumption, from (3.3.12.5) it follows that p and q are unique. Therefore, we can define the following function

$h : W \rightarrow V \times V$,

$h(\langle a,b,t \rangle) = \langle p,q \rangle$, iff $\{p,q\} = [p,q]_M \subseteq [a,b]_M$ and

$d(a,b) \leq t \leq d(a,q)$.

Without loss of generality let $h(\langle a,a,t \rangle) = \langle a,a \rangle$.

If $h(\langle a,b,t \rangle) = \langle p,q \rangle$, then $[p,q]_M$ is the smallest simple segment on $[a,b]_M$, which will contain that point in V' on $[a,b]_M$, at distance t from a. Of course this point should be at distance $t - d(a,p)$ from p.

This is achieved by the following equivalence relation:

Let $\langle a,b,t \rangle, \langle c,e,u \rangle \in W$

$\langle a,b,t \rangle$ Eq $\langle c,e,u \rangle$, iff there are $p,q \in V$, such that either:

* $h(\langle a,b,t\rangle) = \langle p,q\rangle$, $h(\langle c,e,u\rangle) = \langle p,q\rangle$ and $t - d(a,p) = u - d(c,p)$, or
* $h(\langle a,b,t\rangle) = \langle p,q\rangle$, $h(\langle c,e,u\rangle) = \langle q,p\rangle$ and $t - d(a,p) = d(c,p) - u$.

Evidently Eq is reflexive and symmetric by definition.
The transitivity of Eq is straightforward to prove and purely depends on the transitivity of =. Since it is just cumbersome the proof is not performed.
Eq is an equivalence relation.
The equivalence class of $\langle a,b,t\rangle \in W$ is indicated by $[\langle a,b,t\rangle]$.
Note that $[\langle a,b,t\rangle] = [\langle p,q,t-d(a,p)\rangle] = [\langle q,p,d(a,q)-t\rangle]$, where $h(\langle a,b,t\rangle) = \langle p,q\rangle$.
Let $V' = \{[\langle a,b,t\rangle] : \langle a,b,t\rangle \in W\}$.
We will now define the distance function d'.
Let $x = [\langle a,b,t\rangle]$, $y = [\langle c,e,u\rangle] \in V'$, such that $h(\langle a,b,t\rangle) = \langle p,q\rangle$ and $h(\langle c,e,u\rangle) = \langle p',q'\rangle$.

$$d'(x,y) = \begin{cases} |t - u + d(c,p) - d(a,p)| & \text{iff } \langle p,q\rangle = \langle p',q'\rangle \\ |t + u - d(c,p) - d(c,p)| & \text{iff } \langle p,q\rangle = \langle q',p'\rangle \\ \min\{d(p,p') + t - d(a,p) + u - d(c,p'), & \\ \quad d(p,q') + t - d(a,p) + d(c,q') - u, & \\ \quad d(q,p') + d(a,q) - t + u - d(c,p'), & \\ \quad d(q,q') + d(a,q) - t + d(c,q') - u\}, & \\ \quad \text{iff } \{p,q\} \neq \{p',q'\}. & \end{cases}$$

First we have to prove that d' is well defined.
Let $x = [\langle a,b,t\rangle] = [\langle a',b',t'\rangle]$ and
$y = [\langle c,e,u\rangle] = [\langle c',e',u'\rangle]$, such that
$h(\langle a,b,t\rangle) = \langle p,q\rangle$, $h(\langle a',b',t'\rangle) = \langle p',q'\rangle$
$h(\langle c,e,u\rangle) = \langle m,n\rangle$, $h(\langle c',e',u'\rangle) = \langle m',n'\rangle$.
By definition of d' we obviously have
$d'([\langle a,b,t\rangle],[\langle c,e,u\rangle]) =$
$d'([\langle p,q,t-d(a,p)\rangle],[\langle c,e,u-d(c,m)\rangle]) =$
$d'([\langle p',q',t'-d(a',p')\rangle],[\langle c',e',u'-d(c',m')\rangle]) =$
$d'([\langle c,e,u\rangle],[\langle c',e',u'\rangle])$.

Hence, the definition of d' does not depend on the representation of an equivalence class and d' is well defined.

We continue by proving that d' is a distance function.

Clearly $d'(x,x) = 0$ and $d'(x,y) = d'(y,x) \geq 0$ for all $x,y \in V'$.

Let $x,y,z \in V'$. It is sufficient to prove that

* if $d(x,y) = 0$, then $x = y$, and
* $d(x,y) + d(z,y) \geq d(x,z)$.

Let $x = [\langle a,b,t\rangle]$, $h(\langle a,b,t\rangle) = \langle a,b\rangle$,
$y = [\langle c,e,u\rangle]$, $h(\langle c,e,u\rangle) = \langle c,e\rangle$,
$z = [\langle p,q,v\rangle]$ and $h(\langle p,q,v\rangle) = \langle p,q\rangle$.

If $d(x,y) = 0$, then either:

* $\langle a,b\rangle = \langle c,e\rangle$ and $|t - u| = 0$, or
* $\langle a,b\rangle = \langle e,c\rangle$ and $|t + u| = d(a,b)$, or
* $0 = \min \{ d(a,c) + t + u,$
  $d(a,e) + t - u + d(c,e),$
  $d(b,c) + d(a,b) - t + u,$
  $d(b,e) + d(a,b) - t + d(c,e) - u\}$ and
  $\{a,b\} \neq \{c,e\}$.

This evidently leads to $x = y$.

We will now prove the triangle inequality for x,y and z.

There are nine cases.

<u>Case I</u> $d'(x,y) = |t - u|$ and $\langle a,b\rangle = \langle c,e\rangle$, and
$d'(y,z) = |u - v|$ and $\langle c,e\rangle = \langle p,q\rangle$.

Evidently $\langle a,b\rangle = \langle p,q\rangle$ and $d'(x,z) = |t - v|$

Hence, $d'(x,z) \leq d'(x,y) + d'(y,z)$.

<u>Case II</u> $d'(x,y) = |t - u|$ and $\langle a,b\rangle = \langle c,e\rangle$, and
$d'(y,z) = |v + u - d(q,c)|$ and $\langle c,e\rangle = \langle q,p\rangle$.

Evidently $\langle a,b\rangle = \langle q,p\rangle$ and

$$d'(x,z) = |t + v - d(q,p)| \leq |t - u| + |v + u - d(q,p)| = d'(x,y) + d'(y,z).$$

<u>Case III</u> $d'(x,y) = |t + u - d(c,e)|$ and $\langle a,b\rangle = \langle e,c\rangle$, and
$d'(y,z) = |u - v|$ and $\langle c,e\rangle = \langle p,q\rangle$.

Clearly $\langle a,b\rangle = \langle q,p\rangle$ and

$$
\begin{aligned}
d'(x,z) &= |t + v - d(q,p)| \\
&= |t + v - d(c,e)| \\
&\le |t + u - d(q,p)| + |u - v| \\
&= d'(x,y) + d'(y,z).
\end{aligned}
$$

<u>Case IV</u> $d'(x,y) = |t + u - d(a,b)|$ and $\langle a,b\rangle = \langle e,c\rangle$, and

$d'(y,z) = |u + v - d(c,e)|$ and $\langle c,e\rangle = \langle q,p\rangle$.

Clearly $\langle a,b\rangle = \langle p,q\rangle$ and

$$
\begin{aligned}
d'(x,z) &= |t - v| \\
&\le |t - (d(a,b) - u)| + |d(a,b) - u - v| \\
&= |t + u - d(a,b)| + |v + u - d(c,e)| \\
&= d'(x,y) + d'(y,z).
\end{aligned}
$$

<u>Case V</u> $d'(x,y) = |t - u|$, $\langle a,b\rangle = \langle c,e\rangle$ and $\{p,q\} \ne \{c,e\}$.

Hence, $\{p,q\} \ne \{a,b\}$.

Without loss of generality suppose

$d'(y,z) = d(c,p) + u + v$. Then

$$
\begin{aligned}
d'(x,y) + d'(y,z) &= |t - u| + d(c,p) + u + v \\
&\ge d(a,p) + t + v \\
&\ge d'(x,z).
\end{aligned}
$$

<u>Case VI</u> $\{a,b\} \ne \{c,e\}$, $d'(y,z) = |u - v|$ and $\langle c,e\rangle = \langle p,q\rangle$.

Hence, $\{p,q\} \ne \{a,b\}$.

Without loss of generality suppose

$d'(x,y) = d(a,c) + t + u$. Then

$$
\begin{aligned}
d'(x,y) + d'(y,z) &= d(a,c) + t + u + |u - v| \\
&\ge d(a,p) + t + v \\
&\ge d'(x,z).
\end{aligned}
$$

<u>Case VII</u> $\{a,b\} \ne \{c,e\}$, $d'(y,z) = |u + v - d(c,e)|$ and

$\langle e,c\rangle = \langle p,q\rangle$.

Hence, $\{a,b\} \ne \{p,q\}$.

Without loss of generality suppose

$d'(x,y) = d(a,c) + t + u$. Then

$$
\begin{aligned}
d'(x,y) + d'(y,z) &= d(a,c) + t + u + |u + v - d(c,e)| \\
&\ge d(a,c) + t + d(p,q) - v \\
&\ge d(a,p) + t + d(p,q) - v \\
&\ge d'(x,z).
\end{aligned}
$$

<u>Case VIII</u> $d'(x,z) = |t + u - d(c,e)|$, $\langle a,b\rangle = \langle e,c\rangle$ and

$\{c,e\} \ne \{p,q\}$.

Hence, $\{a,b\} \ne \{p,q\}$.

Without loss of generality suppose

$d'(y,z) = d(c,p) + u + v$. Then

$$\begin{aligned} d'(x,y) + d'(y,z) &= |t + u - d(c,e)| + d(c,p) + u + v \\ &\geq d(c,p) + |t - d(c,e)| + v \\ &= d(c,p) + d(c,e) - t + v \\ &= d(b,p) + d(a,b) - t + v \\ &= d(p,b) + d(a,b) - t + v \\ &\geq d'(x,z). \end{aligned}$$

<u>Case IX</u> $\{a,b\} \neq \{c,e\}$ and $\{p,q\} \neq \{c,e\}$.

Without loss of generality suppose

$d'(x,y) = d(a,c) + t + u$ and $d'(y,z) = d(c,p) + u + v$.

Then $d'(x,y) + d'(y,z) = d(a,c) + d(a,p) + t + 2u + v$

$\geq d(a,p) + t + v$.

Since $t + v \geq |t - v|$.

It follows that $d(a,p) + t + v \geq d'(x,z)$.

Hence, $d'(x,y) + d'(y,z) \geq d'(x,z)$.

By this it is proved that $M' = \langle V',d'\rangle$ is a metric space.

To finish this proof it is necessary and sufficient to prove that (3.3.13.1) up to (3.3.13.4) hold for M' and M.

<u>Proof of (3.3.13.1)</u>

First the injective function g is constructed.

$g : V \to V'$ defined by

$c \to [\langle c,c,0\rangle]$

g is clearly a function.

<u>g is injective</u>. Let $g(c) = g(e)$. Then $[\langle c,c,0\rangle] = [\langle e,e,0\rangle]$.

Hence, $\langle c,c,0\rangle$ Eq $\langle e,e,0\rangle$. This leads to the existence of $p,q \in V$, such that either $h(\langle c,c,0\rangle) = \langle p,q\rangle = h(\langle e,e,0\rangle)$ and $d(c,p) = d(e,p) = 0$, or $h(\langle c,c,0\rangle) = \langle p,q\rangle$ and $h(\langle e,e,0\rangle) = \langle q,p\rangle$ and $d(c,p) = d(e,p) = 0$.

From the definition of h it follows that $\{p,q\} = \{c\} = \{e\}$.

Hence, $c = e$, which proves the injectivity.

<u>g preserves the distance d</u>. Let $a,b \in V$.

Then $d'(g(a),g(b)) = d'([\langle a,a,0\rangle],[\langle b,b,0\rangle]\rangle$

$= d(a,b)$.

To complete (3.3.13.1) it is to proved that M' is a full metric space.

Let $t \in d'(V'xV')$, $x,y \in V'$ and $d'(x,y) \geq t$.

It is sufficient to prove that there is a $z \varepsilon [x,y]_{M'}$ such that $d'(x,z) = t$.

Let $x = [\langle a,b,u\rangle]$, $y = [\langle c,e,s\rangle]$ and furthermore, let $h(\langle a,b,u\rangle) = \langle p,q\rangle$ and $h(\langle c,e,s\rangle) = \langle p',q'\rangle$.

There are two cases:

Case 1 $\{p,q\} = \{p',q'\}$.

Without loss of generality suppose $p = p'$ and $q = q'$.

Then $x = [\langle p,q,u'\rangle]$ and $y = [\langle p',q',s'\rangle]$ and $d(x,y) = |u' - s'| \geq t$.

Again without loss of generality suppose $u' \leq s'$.

Then $s' - u' - t \varepsilon d(VxV)$.

Hence, by (3.3.15.2), $||s' - u' - t| - s'| \varepsilon d(VxV)$.

So $u' + t \varepsilon d(VxV)$. Take $z \varepsilon [\langle p,q,u' + t\rangle]$.

Then $d(y,z) = |s' - u' - t| = s' - u' - t$ and

$d'(z,x) = |u' + t - u'| = t$.

Hence, $z \varepsilon [x,y]_{M'}$ and $d'(x,z) = t$.

Case 2 $\{p,q\} \neq \{p',q'\}$. Without loss of generality suppose $d'(x,y) = d(p,p') + u' + s'$.

We have three subcases.

Case 2^A $t \leq u'$. Take $z = [\langle p,q,u' - t\rangle]$.

Then $d'(x,z) = |(u' - t) - u'| = t$ and

$d'(z,y) \leq d(p,p') + (u' - t) + s'$.

From the triangle inequatility it follows:

$z \varepsilon [x,y]_{M'}$ and $d'(x,z) = t$.

Case 2^B $t \geq d(p,p') + u'$. Take $z = [\langle p',q',t - d(p,p') - u'\rangle]$.

Then $d(y,z) = (s' + u' + d(p,p') - t)$ and

$d(z,x) \leq d(p,p') + t - d(p,p') - u' + u' = t$.

From the triangle inequality it follows:

$z \varepsilon [x,y]_{M'}$ and $d'(x,z) = t$.

Case 2^C $0 < t - u < d(p,p')$. Hence, there are $\hat{p}$, $\hat{q}$ $\varepsilon [p,p']_M$, such that $[\hat{p},\hat{q}]_M = \{\hat{p},\hat{q}\}$ and $d(p,\hat{p}) \leq t - u' \leq d(\hat{q},p)$.

Take $z = [\langle \hat{p},\hat{q},t - u' - d(p,\hat{p})\rangle]$.

Then $d'(x,z) \leq d(p,\hat{p}) + u' + t - u' - d(p,\hat{p}) = t$ and

$d'(y,z) \leq d(p,\hat{q}) + d(\hat{p},\hat{q}) - t + u' + d(p,\hat{p}) + s'$

$\leq d(p,\hat{p}) + u' + s' - t$.

Again by the triangle inequality we have:

$z \varepsilon [x,y]_M$ and $d'(x,z) = t$.

This completes the proof of (3.3.13.1).

Proof of (3.3.13.2) Let $x \in V$, $x = [\langle a,b,t\rangle]$. Without loss of generality suppose $h(\langle a,b,t\rangle) = \langle a,b\rangle$.

Obviously $x \in [[\langle a,a,0\rangle],[\langle b,b,0\rangle]]_{M'}$.

Hence, by (3.3.13.3), which we are going to prove next $[\langle a,a,0\rangle],[\langle b,b,0\rangle]]_{M'}$ is simple and we are done.

Proof of (3.3.13.3) Let $a,b \in V$, such that $g(a) = [\langle a,a,0\rangle]$ and $g(b) = [\langle b,b,0\rangle]$.

It is sufficient to prove:

$[a,b]_M$ is simple, iff $[g(a),g(b)]_{M'}$ is simple.

By the preservation of the distance d in M' of (3.3.12.1) the proof of "if" is obvious.

(only if) Suppose $[a,b]_M$ is simple. Let $x \in [g(a),g(b)]_{M'}$, such that $x = [\langle p,q,t\rangle]$ and without loss of generality let $h(\langle p,q,t\rangle) = \langle p,q\rangle$.

It is sufficient to prove:

$\{z \in [g(a),g(b)]_{M'} : d'(g(a),z) \le d'(g(a),x)\} \subseteq [g(a),x]_{M'}$

Without loss of generality suppose $d'(x,g(a)) = d(p,a) + t$.

There are two cases.

Case 1 $d'(x,g(b)) = d(p,b) + t$.

$$\begin{aligned} \text{Hence, } d(a,b) &= d'(g(a),x) + d'(x,g(b)) \\ &= 2t + d(a,p) + d(b,p) \\ &\ge 2t + d(a,p). \end{aligned}$$

So $p \in [a,b]_M$ and $t = 0$.

Take without loss of generality $p = q$ and $t = 0$.

Case 2 $d'(x,g(b)) = d(q,b) + d(p,q) - t$.

Hence, $p,q \in [a,b]_M$.

So $p,q \in [a,b]_M$, $d'(g(a),x) = d(p,a) + t$, and $d'(g(b),x) = d(q,b) + d(p,q) - t$.

Hence, $d(a,p) \le d(a,q)$.

Let $z \in [g(a),g(b)]_{M'}$.

Then similarly there are $\hat{p},\hat{q} \in [a,b]_M$.

So $d'(g(a),z) = d(\hat{p},a) + \hat{t}$,

$d'(g(b),z) = d(\hat{q},b) + d(\hat{p},\hat{q}) - \hat{t}$,

$d(a,\hat{p}) \le d(a,\hat{q})$ and $z = [\langle \hat{p},\hat{q},\hat{t}\rangle]$.

Suppose $d'(g(a),z) \le d'(g(a),x)$.

It is proved that $d'(x,z) + d'(z,g(a)) = d'(x,g(a))$.

Note that either $d(a,\hat{q}) \le d(a,p)$ or $p = \hat{p}$ and $q = \hat{q}$.

If $p = \hat{p}$ and $q = \hat{q}$, then $d'(x,z) = |t - \hat{t}| = t - \hat{t}$.

So $d'(z,g(a)) = \hat{t} + d(a,p)$ and

$d'(x,z) + d'(z,g(a)) = d'(x,g(a))$.

If $d(a,\hat{q}) \leq d(a,p)$, then $d'(x,z) \leq d(p,\hat{q},) + d(\hat{p},\hat{q}) - \hat{t} + t$.

$d'(z,g(a)) = d(\hat{p},a) + \hat{t}$.

Hence, $d'(x,g(a)) \leq d'(x,z) + d'(z,g(a))$

$\leq d(p,\hat{q},) + d(\hat{p},\hat{q}) + d(\hat{p},a) + t$

$= d(a,p) + t$.

This completes the proof of (3.3.13.3).

<u>Proof of (3.3.13.4)</u> Let $a,b \in V$ and let $[g(a),g(b)]_{M'}$ be simple.

Let $x,y \in V'$, such that $x \in [g(a),g(b)]_{M'}$ and

$y \notin [g(a),g(b)]_{M'}$.

Then there are $p,q,\hat{p},\hat{q} \in V$, such that $x = [\langle p,q,t\rangle]$,

$y = [\langle \hat{p},\hat{q},\hat{t}\rangle]$, for some t and $\hat{t}$, $h(\langle p,q,t\rangle) = \langle p,q\rangle$, and

$h(\langle \hat{p},\hat{q},\hat{t}\rangle) = \langle \hat{p},\hat{q}\rangle$.

Obviously $\{p,q\} \neq \{\hat{p},\hat{q}\}$.

Without loss of generality it holds, that

$d'(x,y) = d(p,\hat{p}) + t + \hat{t}$.

So $d'(x,g(p)) \leq d(p,p) + t$ and

$d'(y,g(p)) \leq d(p,\hat{p}) + \hat{t}$.

Thus $d'(x,y) \leq d'(x,g(p)) + d'(y,g(p))$

$\leq d(p,\hat{p}) + t + \hat{t}$

$= d(x,y)$.

Therefore $g(p) \in [x,y]_{M'}$.

Obviously $g(p) \in [a,b]_{M'}$.

This completes the proof.

■

§ 3.4 Continuity and discrete metric spaces.

In this section continuous functions from a discrete metric space to another are studied. Because of the non-standard approach of topologies on discrete metric spaces of the foregoing sections this study becomes non trivial. On the other hand, such continuous functions are more or less characterized by a mesh perturbationally robustness condition. This means that a deviation of size meshwidth in the originals causes a deviation in their corresponding images not greater than the meshwidth. For a full discrete original metric space this property is equivalent to a non-expansiveness property.

This last property which gives rise to a new and stronger class of impossibility theorems, which will be an issue of chapter 4. Therefore they are studied here extensively in connection with the standard topological property of continuity, because continuity is a more natural and better interpretable condition than is non-expansiveness.

First some properties for a function between metric space are defined. Then it is shown that continuity implies mesh perturbationally robustness and conversely a mesh perturbationally robust function guarantees the existence of a continuous function, whenever its domain and codomain are discrete metric spaces. After this, mesh perturbationally robustness and continuity are compared with non-expansiveness.

Let us begin with continuity. Since it is defined in a standard way, only the definition is recalled, without any further comment.

Definition 3.4.1 Continuous

Let F be a function from the set V to the set W, let τ^1 be a topology on V and τ^2 a topology on W.
F is a continuous function with respect to the topologies τ^1 τ^2 (notation (τ^1,τ^2)-continuous), iff for all $O \in \tau^2$: $F^{-1}(O) \in \tau^1$.

■

The following property for functions between metric spaces is called mesh perturbationally robustness. It is defined as follows:

Definition 3.4.2 Mesh Perturbationally Robust

Let F be a function from set V_1 to set V_2 and let $M_1 = \langle V_1, d_1 \rangle$ and $M_2 = \langle V_2, d_2 \rangle$ be two metric spaces. Furthermore, let $mesh_1 := mesh(V_1, d_1)$ and $mesh_2 := mesh(V_2, d_2)$.

F is mesh perturbationally robust, iff for all $x, y \in V_1$: if $d_1(x,y) = mesh_1$, then $d_2(F(x),F(y)) \leq mesh_2$.

■

A function F is mesh perturbationally robust, iff a deviation in the original space V_1 of size $mesh_1$ causes a perturbation in the image space of a size less than or equal to $mesh_2$. Hence, a smallest possible deviation in the original space does not cause a perturbation in the image space, which is greater than the smallest possible deviation in the image space.

Note, if $mesh_1 = 0$, then F is mesh perturbationally robust. If $mesh_2 = 0$ and $mesh_1 > 0$, then F is mesh perturbationally robust, iff for all x,y $F(x) = F(y)$, if $x \sim y$. Here $x \sim y$, iff there are $z_0, z_1, \ldots\ldots, z_k$, such that $z_0 = x$, $z_k = y$ and $d_1(z_i, z_{i+1}) \leq mesh_1$ for all $0 \leq i < k$. Hence, mesh perturbationally robustness is an interesting property only if $mesh_1 > 0$, $mesh_2 > 0$ and $mesh_1 \in d_1(VxV)$.

Example 3.4.3

Let $M_1 = \langle ZxZ, d_1 \rangle$ and $M_2 = \langle ZxZ, d_2 \rangle$, where for all $\langle a,b \rangle, \langle x,y \rangle \in Z \times Z$,

$d_1(\langle a,b \rangle, \langle x,y \rangle) := \sqrt{(a - x)^2 + (b - y)^2}$, and

$d_2(\langle a,b \rangle, \langle x,y \rangle) := |a - x| + |b - y|$.

Let $F : Z \times Z \to Z \times Z$

$\langle a,b \rangle \to \langle a,b \rangle$.

Clearly F is mesh perturbationally robust, although $d_2(F(\langle 1,1 \rangle), F(\langle 0,0 \rangle)) = 2 > \sqrt{2} = d_1(\langle 1,1 \rangle, \langle 0,0 \rangle)$.

Hence, although F does not expand mesh-distant points, it may expand pairs of points with larger distances.

■

Example 3.4.3 leads to the following notion.

Definition 3.4.4 non-expansiveness

Let F be a function from V_1 to V_2 and let $M_1 = \langle V_1,d_1\rangle$ and $M_2 = \langle V_2,d_2\rangle$ be two metric spaces.

F is non-expansive with respect to d_1 and d_2 (notation (d_1,d_2)-non expansive), iff for all $x,y \in V_1$:

$d_1(x,y) \geq d_2(F(x),F(y))$.

■

The notion of non-expansiveness is trivially explained by the formula above. It is evident that non-expansiveness implies mesh perturbationally robustness, whenever the meshwidth, corresponding with the space of originals, is equal to or smaller than the meshwidth, corresponding with the space of images. The reverse is not true as we see in example 3.4.3.

Now we are ready for the first step in the characterization of continuous functions.

Theorem 3.4.5

Let $M_1 = \langle V_1,d_1\rangle$ and $M_2 = \langle V_2,d_2\rangle$ be metric spaces.

Furthermore, let F be a (τ_{d_1},τ_{d_2})-continuous function from $V_1 \cup E_{M_1}$ to $V_2 \cup E_{M_2}$, such that $F(V_1) \subseteq V_2$.

Then $F_{|V_1}$ is mesh perturbationally robust.

Proof of theorem 3.4.5

Let $x,y \in V_1$. By assumption $F(x),F(y) \in V_2$.

Suppose $d_1(x,y) = mesh_1 := mesh(V_1,d_1)$.

It is sufficient to show that $d_2(F(x),F(y)) \leq mesh_2$, where $mesh_2 = mesh(V_2,d_2)$.

Let $O_x = B(F(x),\ mesh_2,\ V_2 \cup E_{M_2},\ d_2)$ and

$O_y = B(F(y),\ mesh_2,\ V_2 \cup E_{M_2},\ d_2)$

Clearly $O_x \in \tau_{d_2}$ and $O_y \in \tau_{d_2}$.

Hence, $F^{-1}(O_x) \in \tau_{d_1}$ and $F^{-1}(O_y) \in \tau_{d_1}$,

since F is $(\tau_{d_1},\ \tau_{d_2})$-continuous.

But because $x \in F^{-1}(O_x)$ and $y \in F^{-1}(O_y)$ there are mesh-edged-balls around x and y contained in $F^{-1}(O_x)$ and $F^{-1}(O_y)$.

So $\{x,y\} \in F^{-1}(O_x) \cap F^{-1}(O_y)$, since $d(x,y) = mesh_1$.

Hence, $F(\{x,y\}) \varepsilon O_x \cap O_y$.
Then $O_x \cap O_y \neq \varphi$.
Hence, either $F(x) = F(y)$ ór $d_2(F(x),F(y)) = mesh_2$.

■

Now we will prove the following. If F is a mesh perturbationally robust function from V_1 to V_2, then we can construct a (τ_{d_1},τ_{d_2})-continuous function $\tilde{F}$ from $V_1 \cup E_{M_1}$ to $V_2 \cup E_{M_2}$, which is an extension of F.

<u>Theorem 3.4.6</u>

Let $M_1 = \langle V_1,d_1\rangle$ and $M_2 = \langle V_2,d_2\rangle$ be metric spaces, such that $mesh_1 := mesh(V_1,d_1) > 0$. Furthermore, let F be a mesh perturbationally robust function from V_1 to V_2.
Then
$\tilde{F}$ from $V_1 \cup E_{M_1}$ to $V_2 \cup E_{M_2}$ is (τ_{d_1}, τ_{d_2})-continuous, where

$$\tilde{F}(\{x,y\}) := \begin{cases} \{F(x),F(y)\} \text{ iff } \{F(x)(F(y)\} \varepsilon E_{M_2} \\ z \text{ iff } \{F(x),F(y)\} \not\varepsilon E_M, \text{ where z is an arbitrary chosen fixed element in } [F(x),F(y)]_{M_2}. \end{cases}$$

Again $\{x,x\}$ is identified with x for all $x \varepsilon V_1$.

<u>Proof of theorem 3.4.6</u>

Let $a \varepsilon V_2$ and $O_a := B(a,mesh_2,E_{M_2} \cup V_2,d_2)$
Suppose $x \varepsilon V_1$, such that $F(x) = a$, and $y \varepsilon V_1$, such that $d_1(x,y) = mesh_1$.
It is sufficient to show that $\tilde{F}(\{x,y\}) \varepsilon O_a$.
By the mesh perturbationally robustness of F we have $d_2(F(y),F(x)) \leq mesh_2$.

<u>Case 1</u> $d_2(F(y),F(x)) = 0$.
Then $F(y) = F(x) = a \varepsilon O_a$ and $\tilde{F}(\{x,y\}) = a$.

<u>Case 2</u> $d_2(F(y),F(x)) = mesh_2 > 0$.
Obviously $\{F(x)(F(y)\} \varepsilon E_{M_2} \cap O_a$ and
$\tilde{F}(\{x,y\}) = \{F(x)(F(y)\} \varepsilon O_a$.

■

Let us give an example how to use the foregoing theorems.

<u>Example 3.4.7</u>

Let $d: R \times R \rightarrow R$ be the Euclidean distance on R given by:

$\langle x,y\rangle \rightarrow |x - y|$.

Let $M_1 = \langle N,d\rangle$ and $M_2 = \langle R,d\rangle$.

Clearly there is no non-constant mesh perturbationally robust function from N to R. It follows then from theorem 3.4.5 that there is no non-constant (τ_d,τ_d)-continuous function F from $N \cup E_{M_1}$ to $R \cup E_{M_2}$, such that $F(N) \subseteq R$.

Let $M_3 = \langle V,d\rangle$, where $V = \{n - 1/_n : n \in \{1,2,3,4.....\}\}$.

$E_{M_3} = \{\{n - 1/_n, n + 1 - 1/_{(n+1)}\} : n \in \{1,2,3,4.....\}\}$.

$mesh_3 := mesh(V,d) = 1 \notin d(V\times V)$.

Hence, there is no non-constant mesh perturbationally robust function from N to V.

Hence, there is no non-constant (τ_d,τ_d)-continuous function F from $N \cup E_{M_1}$ to $V \cup E_{M_3}$, such that $F(N) \subseteq V$.

■

In example 3.4.3 it is shown that in general non-expansiveness is not equivalent to mesh perturbationally robustness even if the meshes are the same. The following theorem deals with this comparison.

<u>Theorem 3.4.8</u>

Let $M_1 = \langle V_1,d_1\rangle$ be a full metric space, such that $mesh(V_1,d_1) > 0$, and $M_2 = \langle V_2,d_2\rangle$ be a metric space, such that $mesh(V_2,d_1) = mesh(V_1,d_1)$. Let F be a function from V_1 to V_2.

Then F is mesh perturbationally robust, iff F is (d_1,d_2)-non-expansive.

<u>Proof of theorem 3.4.8</u>

(if) This is simple to prove.

(only if) Let $x,y \in V_1$, and let $mesh_1 := mesh(V_1,d_1) > 0$.

By theorem 3.3.4 there are $z_0,z_1,z_2,....,z_k \in [x,y]_{M_1}$, such that $z_0 = x$, $z_k = y$ and for all $i \in \{0,1,2,...,k-1\}$ $d_1(z_i,z_{i+1}) = mesh_1$.

It is sufficient to prove that $d_2(F(x),F(y)) \le k \cdot mesh_2$.
By the mesh perturbationally robustness it follows that $d_2(F(z_i),F(z_{i+1})) \le mesh_2$ for all $i \in \{0,1,2,...,k-1\}$.
Hence, by the triangle inequality it is obvious that

$$k \cdot mesh_2 \ge d_2(F(x),F(z_1))+d_2(F(z_1),F(z_2))+...+d_2(F(z_{k-1}),F(y)) \ge d_2(F(x),F(y)).$$

This completes the proof.

■

Using the foregoing theorem we obtain:

Corollary 3.4.9

Let $M_1 = \langle V_1,d_1 \rangle$ and $M_2 = \langle V_2,d_2 \rangle$ be a metric space, such that M_1 is full, $mesh(V_1,d_1) > 0$ and $mesh(V_2,d_2) = mesh(V_1,d_1)$.
Then

3.4.9.1 if F is a (τ_{d_1},τ_{d_2})-continuous function from $V_1 \cup E_{M_1}$ $V_2 \cup E_{M_2}$, such that $F(V_1) \subseteq V_2$, then F is (d_1,d_2)-non-expansive from V_1 to V_2,

3.4.9.2 if F is a (d_1,d_2)-non-expansive function from V_1 to V_2, then there is an (τ_{d_1},τ_{d_2})-continuous function $\tilde{F}$ from $V_1 \cup E_{M_1}$ to $V_2 \cup E_{M_2}$, such that $\tilde{F}(V_1) \subseteq V_2$, for all $\{x,y\} \in V_1 \cup E_{M_1}$.

Proof of corollary 3.4.9

(3.4.9.1) is an immediate consequence of theorem 3.4.5 and theorem 3.4.8.

(3.4.9.2) By theorem 3.4.6 and theorem 3.4.8 it is sufficient to prove that $\tilde{F}$ is well defined.
Let $\{x,y\} \in E_{M_1} \cup V_1$.
It is sufficient to prove that $\{F(x),F(y)\} \in V_2 \cup E_{M_2}$.
This is obviously true if $F(x) = F(y)$.
Suppose $F(x) \ne F(y)$.
Then $x \ne y$ and $\{x,y\} \in E_{M_1}$.
Since M_1 is full $d_1(x,y) = mesh_1 := mesh(V_1,d_1)$.

Hence, by the non-expansiveness of F, it holds that $0 < d_2(F(x),F(y)) \leq d_1(x,y) = mesh_1 \leq mesh_2 := mesh(V_2,d_2)$. So $d_2(F(x),F(y)) = mesh_2$ and therefore $\{F(x),F(y)\} \in E_{M_2}$.

■

Corollary 3.4.9 shows that continuity on discrete domains can have a restricted form, that is, a continuous function is equivalent to a non-expansive one, if the meshwidth of the domain and the range are equal. Although in general non-expansive functions are often continuous, the reverse is seldomly true. However, with this non-standard approach of topologies it follows by corollary 3.4.9 that only non-expansive functions are continuous, whenever the domain and codomain have some special properties.

Note that $\tilde{F}$ is a natural extension of F from V_1 to $V_1 \cup E_{M_1}$.

The non-expansiveness property appears to be very operational and continuity is a property, which is more or less interpretable. The first fact will be shown later on. Let us dwell upon the latter. Let $M_1 = \langle V_1,d_1\rangle$ and $M_2 = \langle V_2,d_2\rangle$ be metric spaces. Then E_{M_1} consists of those pair $\{x,y\}$, such that there is no $z \in V_1 - \{x,y\}$, with $z \in [x,y]_{M_1}$. So, $\{x,y\} \in E_{M_1}$ means that y is a neighbour of x. E_{M_1} contains all this information of points being each others neighbours. Now $O \in \tau_{d_1}$, iff for all $x \in V_1$: if $x \in O$, then $\{x,y\} \in O$ for all $\{x,y\} \in E_{M_1}$, such that $d_1(x,y) = mesh_1 := mesh(V_1,d_1)$. Hence, $O \in \tau_{d_1}$, iff for each point $x \in V_1$ all the information of its closest neighbours is in O.

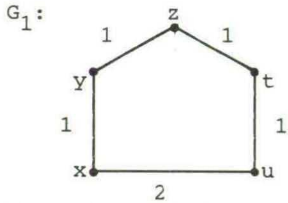
Now a function F from $V_1 \cup E_{M_1}$ to $V_2 \cup E_{M_2}$, such that $F(V_1) \subseteq V_2$, is continuous, iff every open set $O \in \tau_{d_2}$ has an open inverse, that is $F^{-1}(O) \in \tau_{d_1}$. Hence, F is continuous, iff the information of closest neighbours of each point $x \in V_1 \cup F^{-1}(O)$ in M_1 is mapped onto the information of closest neighbours of $F(x) \in V_2$ in M_2. Hence, F preserves the nearby structure of M_1 in M_2.

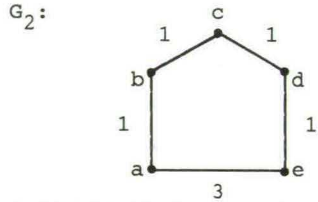
In general as is mentioned before continuity does not imply non-expansiveness. The following example clarifies this fact.

<u>Example 3.4.10</u> <u>Continuous expansive functions</u>

Let $M_1 = \langle V_1,d_1\rangle$ and $M_2 = \langle V_2,d_2\rangle$ be metric spaces, with neighbourhood graphs (see 3.2.5) $G_1 = \langle V_1,E_{M_1}\rangle$ and $G_2 = \langle V_2,E_{M_2}\rangle$.

Let G_1 and G_2 have the following pictures:





Here the numbers at the edges indicate their length.

These pictures determine M_1 and M_2 completely.

Take: $F : V_1 \cup E_{M_1} \rightarrow V_1 \cup E_{M_2}$, such that

$F(x) = a$, $F(y) = b$, $F(z) = c$, $F(t) = d$, $F(u) = e$ and

$F(\{p,q\}) = \{F(p),F(q)\}$ for all $\{p,q\} \in E_{M_1}$.

Obviously F is continuous, but not non-expansive.

■

Since we often use metric spaces, which are not full, we have to do some investigations about non-expansive functions with respect to continuity. §3.3 is now useful.

The following theorem shows the continuity property, which implies non-expansiveness.

<u>Theorem 3.4.11</u>

Let $M_1 = \langle V_1,d_1\rangle$ and $M_2 = \langle V_2,d_2\rangle$ be metric spaces, such that $0 < mesh_1 := mesh(V_1,d_1) = mesh_2 := mesh(V_2,d_2)$ and M_1 has a full image. Then there exists a regular full extension of M_1, say $\hat{M}_1 = \langle \hat{V}_1,\hat{d}_1\rangle$, such that $V_1 \subseteq \hat{V}_1$. Let furthermore, $\hat{F}$ be a $(\tau_{\hat{d}_1},\tau_{d_2})$-continuous function from $\hat{V} \cup E_{\hat{M}_1}$ to $V_2 \cup E_{M_2}$, such that $\hat{F}(V_1) \subseteq V_2$.

Then $F := \hat{F}|_{V_1}$ is (d_1,d_2)-non-expansive.

Proof of theorem 3.4.11

It is obvious that M_1 is weakly full because of the fact that $mesh_1 > 0$ and M_1 has a full image. By theorem 3.3.16 there is a regular full extension of M_1, say $\hat{M}_1 = \langle \hat{V}_1, \hat{d}_1 \rangle$. Without loss of generality suppose that $V_1 \subseteq \hat{V}_1$. From the construction in theorem 3.3.16 it follows immediately that $mesh(\hat{V}_1, \hat{d}_1) = mesh_1$. Hence, by corollary 3.4.9 the conclusion of the theorem follows.

■

Theorem 3.4.11 shows that continuity on regular full extensions implies non-expansiveness, if the meshes are equal and positive. Let us end this section by an explanation of this implication. In general the continuity property does not take the length of edges into account. The non-expansiveness is strongly related to these lengths. Hence, in general continuity does not imply non-expansiveness. In a full metric space, however, the length of an edge is equal to the meshwidth of the space, therefore continuity implies non-expansiveness in these spaces, whenever the meshes are equal and positive. If we demand continuity on a regular full extension, then we add this information about the length of an edge in the original space to our system. In that case continuity is able to distinguish the length of the original edge and non-expansiveness can therefore be deduced from the continuity property.

§ 3.5 Background for impossibility theorems

This section is devoted to a development of mathematical theorems by which several new impossibility theorems based on non-expansiveness are proved in the next chapter. The tools, which are introduced here, may be useful in other parts of science, but the author has only searched for applications in the theory of social choice. Since a considerable amount of applications are found there, the theory of this section became important enough to develop it seperately here.

In this section metric spaces and non-expansive functions are studied. It appears that under these functions some structural properties of the metric spaces become invariant. To be more specific. Let F be a non-expansive function from metric space $M_1 = \langle V_1,d_1\rangle$ to $M_2 = \langle V_2,d_2\rangle$. Suppose that certain conditions concerning F, M_1 and M_2 are fulfilled (which of their technical description are not mentioned here). Then: if $x \in V_1$ lies on the neighbourhood of a special type of division of M_1, then $F(x) \in V_2$ is on the neighbourhood of a similar type of division of M_2. The usefulness of this result may be clear, when the reader is told, that there is a correspondence between sets of points in V_i and sets of divisions of M_i. Hence only a few possible points can be assigned by F to a specific point $x \in V_1$. It is this restriction of the possibilities for F(x), which enables us to prove impossibility theorems for non-expansive welfare functions.

First some new notions are introduced.

<u>Defition 3.5.1</u>

Let $M = \langle V,d\rangle$ be a metric space, with meshwidth mesh.
Furthermore, let $x \in V$, $u \in R$, $u \geq 0$ and let X,Y,L and R be subsets of V.
Then

3.5.1.1 (a) x <u>covers</u> X <u>within radius</u> u, iff $X \subseteq B(x,u,V,d)$.

(b) $\text{Cover}_M(X,u) := \{x \in V : x \text{ covers } X \text{ within radius } u\}$.

(c) Y <u>covers</u> X <u>within radius</u> u, iff $Y \subseteq \text{Cover}_M(X,u)$.

3.5.1.2 x is <u>in the neighbourhood of</u> L, iff for all t > mesh there is a $y \in L$, such that $y \in B(x,t,V,d)$.

3.5.1.3 x is <u>in the neighbourhood of</u> <L,R>, iff for all t > mesh there are y_L ε L and y_R ε R, such that y_L, y_R ε B(x,t,V,d).

3.5.1.4 NH_M(L) := {x ε V : x is in the neighbourhood of L} is <u>the neighbourhood of</u> L.

3.5.1.5 NH_M(L,R) := {x ε V : x is in the neighbourhood of <L,R>} is <u>the neighbourhood of</u> <L,R>.

■

x covers X within radius u iff every point y in X can be reached from x within distance u. $Cover_M$(X,u) is the set of all those points x. Y covers X within radius u, whenever for all y ε Y, y covers X within radius u.

x is in the neighbourhood of L, iff the distance between x and L is as small as possible in M. NH_M(L,u) is the set of points which have such a small distance from L.

x is in the neighbourhood of <L,R>, iff the distance between x and L as well as the distance between x and R is as small as possible in M. NH_M(L,R,u) is the set of points whit these properties.

For examples of these notions as well as other notions introduced in this section the reader is referred to §4.4. There also useful applications of these notions as well as the following theorems can be found.

We have the following result.

<u>Theorem 3.5.2</u>

Let M = <V,d> be a metric space, u ≥ 0, R,L $\subseteq$ V, and mesh := mesh(V,d).

Then $NH_M(L,R) = NH_M(L) \cap NH_M(R)$.

<u>Proof of theorem 3.5.2</u>

x ε NH_M(L,R),

iff

for all t > mesh there are y_L ε L and y_R ε R, such that y_R, y_L ε B(x,t,V,d),

iff

for all t > mesh: L ∩ B(x,t,V,d) ≠ φ and R ∩ B(x,t,V,d) ≠ φ,

iff

for all $t >$ mesh: $L \cap B(x,t,V,d) \neq \varphi$ and for all $t >$ mesh $R \cap B(x,t,V,d) \neq \varphi$,

iff

$x \in NH_M(L) \cap NH_M(R)$.

■

By theorem 3.5.2 it follows that the neighbourhood of the neighbourhood of two regions is equal to the intersection of two neighbourhoods of those regions.

Although there are many neighbourhoods, only a few (compared to their total number) are used here.

<u>Definition 3.5.3</u> <u>Standard Neighbourhood</u>

Let $M = \langle V,d\rangle$ be a metric space, let u_1 and u_2 be non-negative real numbers, let X and Y be subsets of V and let $y \in V$.

3.5.3.1 (a) y is <u>in the standard neighbourhood based on</u> $\langle X,u_1\rangle$, iff $y \in NH_M(Cover_M(X,u_1))$.

(b) $SNH_M(X,u_1) := \{x \in V : x \in NH_M(Cover_M(X,u_1))\}$.

3.5.3.2 (a) y is <u>in the standard neighbourhood based on</u> $\langle X,u_1\rangle$ <u>and</u> $\langle Y,u_2\rangle$, iff $y \in NH_M(Cover_M(X,u_1),Cover(Y,u_2))$.

(b) $SNH_M(\langle X,u_1\rangle,\langle Y,u_2\rangle) := SNH_M(\langle X,u_1\rangle) \cap SNH_M(\langle Y,u_2\rangle)$.

■

Now since the cover of a set may be important another description of such sets is introduced.

<u>Definition 3.5.4</u> <u>Circularly enclosedness</u>

Let $M = \langle V,d\rangle$ be a metric space, u a non-negative real number, and X and Y subsets of V.

X <u>circularly encloses</u> Y <u>with diameter</u> u, iff

3.5.4.1 for all $x \in V - X$ there is an $y \in Y$, such that: $d(x,y) \geq u$, and

3.5.4.2 for all $x \in X$ and all $y \in Y$ $d(x,y) < u$.

■

The following result motivates the introduction of circularly enclosedness.

Theorem 3.5.5

Let $M = \langle V,d\rangle$ be a metric space, u a non-negative real number, and $X,Y \subseteq V$.

Then (I) and (II) are equivalent, where

(I) $X = Cover_M(Y,u)$,

(II) X circularly encloses Y with diameter u.

Proof of theorem 3.5.5

Note that $X \subseteq Cover_M(Y,u)$ is equivalent to (3.5.4.2) and $Cover_M(Y,u) \subseteq X$ is equivalent to (3.5.4.1)

■

All the notions introduced above are used to study a special type of non-expansive functions. Now these functions are described.

In the rest of this section the following assumption is often quoted.

Assumptions 3.5.6

3.5.6.1 Let V and W be sets, such that $V \subseteq W$.

3.5.6.2 For $i \in \{0,1,2\}$ let $M_i := \langle V_i,d_i\rangle$ be a metric space, such that $mesh_i := mesh(V_i,d_i)$ and furthermore, suppose:

$V_0 = V$, $V_1 = W$ and $V_2 = V \times V$,

$mesh_0 = mesh_2 \leq mesh_1$ and

for all $\langle a,b\rangle,\langle x,y\rangle \in VxV$ $d_2(\langle a,b\rangle,\langle x,y\rangle) = d_0(a,x)+d_0(y,b)$.

3.5.6.3 Let F be a function from VxV to W, such that:

(i) F is (d_2,d_1)-non-expansive and

(ii) F leaves the diagonal invariant, i.e., for all $\langle a,a\rangle \in VxV : F(\langle a,a\rangle) = a$.

■

Functions for which assumption 3.5.6.3 holds play an important rôle in the next chapter. There it is shown, that the non-expansiveness is a weaker condition than the independence of irrelevant alternatives and the diagonal invariance is equivalent to Pareto-optimality. Hence, we will often consider functions with assumption 3.5.6.3.

Now the following holds:

Lemma 3.5.7

Assume 3.5.6, let $X \subseteq V$ and let u be a non-negative real number.

Then $F(\mathrm{Cover}_{M_2}(\bar{d}(X\times X),u) \subseteq \mathrm{Cover}_{M_1}(X,u)$.

Proof of lemma 3.5.7

Let $z \in X$ and $\langle z_1,z_2\rangle \in \mathrm{Cover}_{M_2}(\bar{d}(X\times X),u)$.

It is sufficient to prove that $d_1(z,F(\langle z_1,z_2\rangle)) < u$.

Note that $u > d_2(\langle z_1,z_2\rangle,\langle z,z\rangle)$

$\geq d_1(F(\langle z_1,z_2\rangle,F(\langle z,z\rangle))$

$= d_1(F(\langle z_1,z_2\rangle),z)$.

■

The following theorem shows that standard neighbourhoods are mapped in standard neighbourhoods by F.

Theorem 3.5.8

Assume 3.5.6, let $X,Y \subseteq V$ and let u_1 and u_2 be non-negative real numbers. Then

3.5.8.1 $F(\mathrm{SNH}_{M_2}(\bar{d}(X\times X),u_1) \subseteq \mathrm{SNH}_{M_1}(X,u_1)$,

3.5.8.2 $F(\mathrm{SNH}_{M_2}(\langle\bar{d}(X\times X),u_1\rangle,\langle\bar{d}(Y\times Y),u_2\rangle) \subseteq$

$\mathrm{SNH}_{M_1}(\langle X,u_1\rangle,\langle Y,u_2\rangle)$.

Proof of theorem 3.5.8

(3.3.8.1) Let $\langle x,y\rangle \in \mathrm{SNH}_{M_2}(\bar{d}(X\times X),u_1)$ and $e > \mathrm{mesh}_1 \geq \mathrm{mesh}_2$.

Hence, there is a $\langle z_1,z_2\rangle \in \mathrm{Cover}_{M_2}(\bar{d}(X\times X),u_1)$ and $d_2(\langle z_1,z_2\rangle,\langle x,y\rangle) < e$.

It is sufficient to prove that $d_1(F(\langle z_1,z_2\rangle),F(\langle x,y\rangle)) < e$ and $F(\langle z_1,z_2\rangle) \in \mathrm{Cover}_{M_1}(X,u_1)$.

The first follows by the non-expansiveness of F and the second by 3.5.7.

(3.5.8.2) Note that $F(\mathrm{SNH}_{M_2}(\langle\bar{d}(X\times X),u_1\rangle,\langle\bar{d}(Y\times Y),u_2\rangle)) =$

$F(\mathrm{SNH}_{M_2}(\bar{d}(X\times X),u_1) \cap \mathrm{SNH}_{M_2}(\bar{d}(Y\times Y),u_2)) \subseteq$

$F(\mathrm{SNH}_{M_2}(\bar{d}(X\times X),u_1)) \cap F(\mathrm{SNH}_{M_2}(\bar{d}(Y\times Y),u_2)) \subseteq$

$\mathrm{SNH}_{M_1}(X,u_1) \cap \mathrm{SNH}_{M_1}(Y,u_2)) = \mathrm{SNH}_{M_1}(\langle X,u_1\rangle,\langle Y,u_2\rangle)$.

■

Theorem 3.5.8 reveals an invariance property of non-expansive functions, which leave the diagonal invariant. This property however is not operational because, we have no regular way how we can deal with standard neighbourhoods. In fact these neighbourhoods may be very difficult to describe. Fortunately we can state sufficient conditions, such that a point $\langle x,y\rangle \in VxV$ is in a standard neighbourhood of a region in VxV. To state these conditions we need another notion.

<u>Example 3.5.9</u> <u>Approximation by ellipses</u>

Let $M = \langle V,d\rangle$ be a metric space, let u be a non-negative real number, let $x,y \in V$ and let $X \subseteq V$.

X <u>can be approximated by interiors of ellipses from</u> x <u>and</u> y <u>and radius</u> u, iff for all $e > \mathrm{mesh}(V,d)$ there are $x',y' \in V$, such that $d(x,x') + d(y,y') < e$, and for all $z \in X$: $d(x',z) + d(z,y') < u$.

■

The notion of definition 3.5.9 is still a technical one, at least to the author. He has not (yet) been able to enrich this notion with more interpretations than those which are pointed out in the name of this notion. Therefore no further discussion is spent on it.

The following theorem indicates the relation between standard neighbourhoods and approximations by interiors of ellipses.

<u>Theorem 3.5.10</u>

Assume 3.5.6, let $X \subseteq V$, let $\langle x,y\rangle \in V \times V$ and let u be a non-negative real number.

Then $\langle x,y\rangle \in SNH_{M_2}(\bar{d}(XxX),u)$, iff X can be approximated by interiors of ellipses from x and y and radius u.

<u>Proof of theorem 3.5.10</u>

$\langle x,y\rangle \in SNH_{M_2}(\bar{d}(XxX),u)$,

iff $\langle x,y\rangle \in NH_{M_2}(Cover_{M_2}(\bar{d}(XxX),u))$,

iff for all $e > mesh_2$ there is an $\langle a,b\rangle \in Cover_{M_2}(\bar{d}(XxX),u)$, such that $d_2(\langle x,y\rangle,\langle a,b\rangle) < e$, iff for all $e > mesh_2 = mesh_0$ there is an $\langle a,b\rangle \in V \times V$, such that for all $\langle z,z\rangle \in \bar{d}(XxX)$ it holds that:

$d_2\langle a,b\rangle,\langle z,z\rangle) < u$ and $d_2\langle x,y\rangle,\langle a,b\rangle) < e$,

iff for all $e > mesh_0$ there are $a,b \in V$, such that for all $z \in X$ it holds that:

$d_0(a,z) + d_0(z,b) < u$ and $d_0(x,a) + d_0(y,b) < e$,

iff X can be approximated by interiors of ellipses from x and y and radius u.

■

We now have as a final result of this chapter.

Corollary 3.5.11

Assume 3.5.6, let $\hat{x}, \hat{y} \in V$, let $L,R \subseteq V$, let $L',R' \subseteq W$ and let u_L, u_R be non-negative real numbers.

Suppose furthermore:

3.5.11.1 L can be approximated by interiors of ellipses from $\hat{x}$ and $\hat{y}$ and radius u_L,

3.5.11.2 R can be approximated by interiors of ellipses from $\hat{x}$ and $\hat{y}$ and radius u_R,

3.5.11.3 L' circularly encloses L with diameter u_L, and

3.5.11.4 R' circularly encloses R with diameter u_R.

Then $F(\hat{x},\hat{y}) \in NH_{M_2}(L',R')$.

Proof of corollary 3.5.11

By theorem 3.5.10, definition 3.5.3.2 and (3.5.11.1) and (3.5.11.2) it follows: $(\hat{x},\hat{y}) \in SNH_{M_2}(\langle L,u_L\rangle,\langle R,u_R\rangle)$.

Hence by (3.5.8) we have:

$$F(\langle\hat{x},\hat{y}\rangle) \in SNH_{M_2}(\langle L,u_L\rangle,\langle R,u_R\rangle) = NH_{M_2}(Cover_M(L,u_L),Cover_M(R,u_R)).$$

By theorem 3.5.5 and (3.5.11.3) and (3.5.11.4) it follows:

$F(\langle\hat{x},\hat{y}\rangle) \in NH_{M_2}(L',R')$.

■

The results of this section are important for several impossibility theorems discussed in the following chapter.

Chapter 4 WELFARE FUNCTIONS

In this chapter social choice problems in a strict sense are studied. In literature Pareto-optimality, neutrality and independence of irrelevant alternatives are often explicitly or implicitly imposed on welfare functions, decision rules that model these problems. In section 4.1 it is shown that welfare functions, satisfying these three conditions, correspond in a specific unique way with order morphisms. To be more precise, the domain and the range of the welfare function is extended and this extended welfare function is an order morphism.

Moreover, it is shown in section 4.2 that choice correpondences, satisfying specific (often explicitly or implicitly imposed) conditions, are reconstructable by orderings. So, we have modeled the social choice problems in terms of order morphisms that is in terms of order preserving mappings.

Since the models in social choice theory can often be described by order morphisms, we can study social choice problems by order morphisms in the framework of the classification system of orderings. In section 4.3, it is shown that the Arrow-paradox occurs, whenever the range of an order morphism (welfare function) satisfies some transitivity conditions. These conditions are very weak; much more weaker than the usually imposed ones. Therefore, this result generalizes many well-known results. Moreover, by the classification mechanism we are no longer obliged to describe the orderings in the range by transitivity conditions. So we are able to describe meaningful sets of orderings, which do not satisfy transitivity or acyclicity conditions, conditions which are often referred to as being the `rationality' of society. In spite of this weakening of the conditions of the range of an order morphism (welfare function), the Arrow-paradox does not disappear. The result is that the Arrow-paradox does not disappear by meaningfully weakenings of the range conditions of an order morphism (welfare function).

In section 4.4 a weakening of the independence of irrelevant alternatives, called continuity, is discussed. It is clear that replacing continuity, instead of the independence conditions, does not garantee that the welfare function is an order morphism. So we have to leave the framework of order morphisms and the classification system, when substituting the independence of irrelevant alternatives by the weaker continuity condition.

Although continuity is a weaker condtion than the independence condition, the Arrow-paradox does not disappear when the latter is replaced by the former.

In the last section it is shown that by restricting the domain of a welare function, it is possible that the Arrow-paradox disappears. Such restrictions can be interpreted as (inter)dependencies between/of the individual preferences.

§ 4.1 Welfare Functions and Order Morphism's

In this section an equivalence is proved between the notion of an order morphism and a welfare function. That is Pareto-optimal, neutral and independent of irrelevant alternatives. This equivalence enables us to study impossibility theorems in the framework of classified sets of orderings, which is done in section 4.3. Before stating the equivalence two problems have to be solved. The first problem is concerned with the domain and range of a welfare function. These two are fixed on finite sets of alternatives, whereas the domain and range of an order morphism vary over any finite subset of the universe U. Hence, we have to extend the notion of welfare function, such that all those subsets are in its domain and its range.

The second problem is concerned with the domain of an order morphism. This is a set of relations, whereas the domain of a welfare function is a set of profiles. Hence, we have to extend the notion of a classifiable set of orderings to a set of profiles. Along with this extension, the domain of an order morphism becomes a set of profiles.

First we cope with the second problem. Before being able to define a classification system for sets of profiles it is inevitable to define several operations on profiles as well as on sets of profiles.

Let Æ, Å and U be, as in chapter 2, the set of possible domains, the set of possible relations and the set of elements, which we are interested in, respectively. Furthermore, let n be a positive integer, then

$$Å_n := \{\langle R_A^1, R_A^2, R_A^2, \ldots, R_A^n\rangle : R_A^k \in Å \text{ for all } k \in \{1,2,3\ldots,n\}\}$$

is the <u>set of possible profiles</u>. Notice that $Å_1 = Å$, when $\langle R_A^1\rangle$ is identified with R_A^1.

Instead of $\langle R_A^1, R_A^2, R_A^2, \ldots, R_A^n\rangle$ a profile in $Å_n$, we also write r_A.

Next the operations permutation, conversion, restriction, concatenation and substitution are extended componentwise to $Å_n$.

Defintion 4.1.1   Operations on profiles

Let $n \geq 1$.

4.1.1 For all profiles $\langle R_A^1, R_A^2, R_A^3, \ldots, R_A^n \rangle = r_A$ in Å and permutations σ is S_U: $\sigma r_A := \langle \sigma R_A^1, \sigma R_A^2, \sigma R_A^2, \ldots, \sigma R_A^n \rangle$ is the permutation of r_A according to σ.

4.1.2 For all profiles $\langle R_A^1, R_A^2, R_A^3, \ldots, R_A^n \rangle = r_A$ in Å : $\bar{v} r_A := \langle \bar{v} R_A^1, \bar{v} R_A^2, \bar{v} R_A^3, \ldots, \bar{v} R_A^n \rangle$ is the converse of r_A.

4.1.3 For all profiles $\langle R_A^1, R_A^2, R_A^3, \ldots, R_A^n \rangle = r_A$ in Å and X ε Æ, such that $X \subseteq A$: $r_{A|_X} := \langle R^1_{A|_X}, R^2_{A|_X}, \ldots, R^n_{A|_X} \rangle$ is the restriction of r_A to X.

4.1.4 For all profiles $\langle R_A^1, R_A^2, R_A^3, \ldots, R_A^n \rangle = r_A$ and $\langle \hat{R}_B^1, \hat{R}_B^2, \hat{R}_B^3, \ldots, \hat{R}_B^n \rangle = \hat{r}_B$ in $Å_n$, such that $A \cap B = \varphi$: $r_A \gg \hat{r}_B := \langle R_A^1 \gg \hat{R}_B^1,\ R_A^2 \gg \hat{R}_B^2, \ldots,\ R_A^n \gg \hat{R}_B^n \rangle$ is the concatenation of r_A with r_B.

4.1.5 For all profiles $\langle R_A^1, R_A^2, R_A^3, \ldots, R_A^n \rangle = r_A$ and $\langle \hat{R}_B^1, \hat{R}_B^2, \hat{R}_B^2, \ldots, \hat{R}_B^n \rangle = \hat{r}_B$ in $Å_n$, such that $A \cap B = \varphi$ and for all $x \in A$:

$Sub(r_A, x, \hat{r}_B) := \langle Sub(R_A^1, x, \hat{R}_B^1), Sub(R_A^2, x, \hat{R}_B^2) \ldots, Sub(R_A^n, x, \hat{R}_B^n) \rangle$

is the substitution of $\hat{r}_B$ in r_A instead of x.

■

Clearly all these operations are extensions of the operations defined in chapter 2 with the same names. Therefore no further comment is spent on these operations.

By virtue of these operations it is possible to extend the notion "classified as set of orderings" to "classified as set of profiles".

Definition 4.1.2   Classification of profiles on orderings

Let V be a set of profiles, i.e. $V \subseteq Å_n$ for some $n \in \{1,2,3,4,..\}$, then V is classified as a set of profiles, iff V is non-trivial, and is closed under permutation, conversion, restriction, concatenation and substitution.

Here V is:

4.1.2.1 <u>closed under permutation</u>, iff $\sigma r_A \in V$, for all $r_A \in V$ and all $\sigma \in S_U$,

4.1.2.2 <u>closed under conversion</u>, iff $\bar{v} r_A \in V$, for all $r_A \in V$,

4.1.2.3 <u>closed under restriction</u>, iff $r_{A|_X} \in V$, for all $r_A \in V$ and all $X \subseteq A$, with $X \neq \varphi$,

4.1.2.4 <u>non-trivial</u>, iff for all $A \in \mathring{A}$ there are $r_A, \hat{r}_A \in \mathring{A}_n$, such that $r_A \in V$ and $\hat{r}_A \notin V$,

4.1.2.5 <u>closed under concatenation</u>, iff $r_A \gg \hat{r}_B \in V$, for all r_A and $\hat{r}_A$ in V, such that $A \cap B = \varphi$, and

4.1.2.6 <u>closed under substitution</u>, iff $Sub(r_A, x, \hat{r}_B) \in V$, for all r_A and $\hat{r}_B$ in V, with $A \cap B = \varphi$ and $\hat{r}_B = \bar{v} r_B$, and all $x \in A$.

■

The introduced notions of definition 4.1.2 are extensions of definitions in chapter 2 with the same name. Note in particular that V is classified as set of orderings, iff V is classified as set of profiles on orderings, whenever $V \subseteq \mathring{A}_1$.

Furthermore, if $V \subseteq \mathring{A}$ is classified as set of orderings, then $V_n := \{\langle R_A^1, R_A^2, R_A^3, \ldots, R_A^n\rangle : R_A^k \in V \text{ for all } k \in \{1, \ldots, n\}\}$ can be classified as sets of profiles. V_n is called the set of profiles on V. The proof is simple and left to the reader. Let $X \in Æ$. Then <u>the set of profiles on</u> V <u>and</u> X is $V_n(X) := \{r_A \in V_n : A = X\}$. Instead of $V_1(X)$, $V(X)$ is often written. In literature only sets of type $V_n(X)$ and $V(X)$ are used.

Now we extend the domain and the range of an order morphism.

<u>Definition 4.1.3</u> <u>Order morphism</u>

Let V, and W be two subsets of $\mathring{A}$, such that $V \subseteq \mathring{A}_n$ and $W \subseteq \mathring{A}_m$ (m and n are positive integers). Furthermore, suppose that V is classified as a set of profiles.

F is an <u>order morphism from</u> V <u>to</u> W, iff F satisfies (4.1.3.1) up to (4.1.3.6).

Here:

4.1.3.1 $F(r_A) \in W(A)$, for all $A \in Æ$ and $r_A \in V$,

4.1.3.2 $F(\sigma r_A) = \sigma F(r_A)$, for all $\sigma \in S_u$ and $r_A \in V$,

4.1.3.3 $F(\bar{v} r_A) = \bar{v} F(r_A)$, for all $r_A \in V$, 4.1.3.4 $F(r_{A|_B}) = F(r_A)|_B$, for all $r_A \in V$ and all $B \subseteq A$, with $B \neq \varphi$,

4.1.3.5 $F(r_A \gg \hat{r}_B) = F(r_A) \gg F(\hat{r}_B)$, for all $r_A, \hat{r}_B \in V$, with $A \cap B = \varphi$, and

4.1.3.6 $F(\mathrm{Sub}(r_A,x,\hat{r}_B)) \doteq \mathrm{Sub}(F(r_A),x,F(\hat{r}_B))$, for all $r_A, \hat{r}_B \in V$, with $A \cap B = \varphi$ and $\hat{r}_B = \bar{v}r_B$, and all $x \in A$.

■

The following theorem holds similarly to theorem 2.2.25.

Theorem 4.1.4

Suppose $V \subseteq Å_n$ and $W \subseteq Å_m$ (m and n are positive integers), V is classified as a set of profiles and F is an order morphism from V to W. Then $F(V) \subseteq W$ is classified as a set of profiles.

■

Up to here only definitions and notions about sets of orderings have been extended to sets of profiles. Now the same remains to be done for welfare functions.

Let $\Gamma = \langle A,N\rangle$ be a society ($|A| \geq 2$ and $|N| = n \geq 1$). Remember that a welfare function F on Γ is a function from a set of profiles on A to a set of orderings on A. If V and W are two sets which are classified as sets of orderings, then a function F from $V_n(A)$ to $W(A)$ is a welfare function on Γ from $V_n(A)$ to $W(A)$. The next definition extends this notion.

Definition 4.1.5 Complete extension

Suppose $\Gamma = \langle A,N\rangle$ is a society, with $|A| \geq 2$ and $|N| = n \geq 1$, and V and W are two sets of relations, where V is classified as set of orderings.

4.1.5.1 A complete welfare function F (on N) from V_n to W is a function from V_n to W, such that for all $X \in Æ$: $F|_{V_n(X)}$ is a welfare function on $\langle X,N\rangle$ from $V_n(X)$ to $W(X)$.

4.1.5.2 Let $\hat{F}$ be a welfare function on Γ from $V_n(A)$ to $W(A)$. $\hat{F}$ is a <u>complete extension of</u> F, iff $\hat{F}$ is a complete welfare function on N from V_n to W and $\hat{F}|_{V_n(A)} = F$.

■

After these extension definitions of welfare functions several (social) conditions for welfare functions have to be reformulated for complete welfare functions.

<u>Definition 4.1.6</u>

Let N = {1,2,3,....,n}, $V \subseteq \mathring{A}$ be classified as a set of ordering and $\hat{F}$ a complete welfare function on N from V_n to $\mathring{A}$. Then $\hat{F}$ is:

4.1.6.1 <u>Pareto-optimal</u>, iff $\bar{a}R_X^1 \subseteq \bar{a}\hat{F}(r_X)$, for all $r_X \in V_n$, with $R_X^i = R_X^j$, for all $i,j \in N$,

4.1.6.2 <u>indepedent of irrelevant alternatives</u>, iff $\hat{F}(r_{X|_Y}) = \hat{F}(r_X)|_Y$, for all $r_X \in V_n$ and all $Y \subseteq X$, with $Y \neq \varphi$,

4.1.6.3 <u>neutral</u>, iff $\hat{F}(\sigma r_X) = \sigma\hat{F}(r_X)$, for all $r_X \in V_n$ and all $\sigma \in S_U$,

4.1.6.4 <u>symmetric</u>, iff $\hat{F}(\bar{v}r_X) = \bar{v}\hat{F}(r_X)$, for all $r_X \in V_n$.

■

The following theorem proves that the properties for complete welfare functions defined as in (4.1.6) are strong enough to imply the equally named properties for welfare functions.

<u>Theorem 4.1.7</u>

Suppose N = {1,2,...,n}, $V \subseteq \mathring{A}$ is classified as a set of orderings, $A \in Æ$ and $\hat{F}$ is a complete welfare function from V_n to $\mathring{A}$. Define F from $V_n(A)$ to $\mathring{A}(A)$ on <A,N> by

$F := \hat{F}|_{V_n(A)}$.

4.1.7.1 If $\hat{F}$ is Pareto-optimal, then F is Pareto-optimal.

4.1.7.2 If $\hat{F}$ is indepedent of irrelevant alternatives, then F is indepedent of irrelevant alternatives.

4.1.7.3 If $\hat{F}$ is neutral, then F is neutral.

4.1.7.4 If $\hat{F}$ is symmetric, then F is symmetric.

Proof of theorem 4.1.7

(4.1.7.1), (4.1.7.3) and (4.1.7.4) are trivial.

(4.1.7.2) Let r_A, $\hat{r}_A$ ε $V_n(A)$ and $B \subseteq A$, with $B \neq \varphi$ and $r_{A|_B} = \hat{r}_{A|_B}$.

It is sufficient to prove $\hat{F}(r_A)|_B = \hat{F}(\hat{r}_A)|_B$.

Since $\hat{F}(r_{A|_B}) = \hat{F}(r_A)|_B$, $\hat{F}(\hat{r}_{A|_B}) = \hat{F}(\hat{r}_A)|_B$ and $r_{A|_B} = \hat{r}_{A|_B}$, this follows evidently.

■

By theorem 4.1.7 it follows that every Pareto-optimal, independent of irrelevant alternatives, neutral or symmetric complete welfare function is a complete extension of a welfare function on a arbitrary chosen society <A,N>, with A ε $\mathcal{E}$, which is Pareto-optimal, independent of irrelevant alternatives, neutral or symmetric respectively. The properties introduced for complete welfare functions are stronger than those for welfare functions. In the latter no constraints are imposed on profiles which do not belong to the domain of a welfare function but still to its complete extension. In the following theorem we show that independent of irrelevant alternatives and neutral welfare functions, can be extended in an unique way to complete independent of irrelevant alternatives and neutral welfare functions.

Theorem 4.1.8 Unique complete extension

Suppose Γ = <A,N> is a society, with $|A| \geq 2$ and $|N| = n$, V is classified as a set of orderings and F : $V_n(A) \to \AA$ a neutral and independent of irrelevant alternatives welfare function on Γ from $V_n(A)$ to $\AA$.

Then there is a unique complete extension $\hat{F}$ of F, which is neutral and independent of irrelevant alternatives.

Proof of theorem 4.1.8

For arbitrary r_X ε V_n and x,y ε X define $\hat{F}$ as follows: <x,y> ε $\hat{F}(\hat{r}_X)$, iff there is a permutation σ ε S_U and a profile r_A ε $V_n(A)$, such that $\sigma(\hat{r}_{X|_{\{x,y\}}}) = r_{A|_{\{\sigma(x),\sigma(y)\}}}$ and <σ(x),σ(y)> ε $F(r_A)$.

By this definition $\hat{F}$ is a complete welfare function, which is independent of irrelevant alternatives and neutral.

To prove that $\hat{F}$ is a complete extension of F take a,b ε A and $\hat{r}_A$ ε $V_n(A)$.

Taking the identity permutation yields:

if <a,b> ε $F(\hat{r}_A)$, then <a,b> ε $\hat{F}(\hat{r}_A)$.

Suppose <a,b> ε $\hat{F}(\hat{r}_A)$. Then there is a σ ε S_U and r_A ε $V_n(A)$, such that $\sigma(\hat{r}_{A}|_{\{a,b\}}) = r_{A}|_{\{\sigma(a),\sigma(b)\}}$ and $\langle\sigma(a),\sigma(b)\rangle$ ε $F(r_A)$.

Take σ' ε S_A, such that $\sigma'|_{\{a,b\}} = \sigma|_{\{a,b\}}$.

Then $\sigma'(\hat{r}_A)$ ε $V_n(A)$ and $\sigma'(\hat{r}_A)|_{\{a,b\}} = \hat{r}_{A}|_{\{\sigma(a),\sigma(b)\}}$.

Now we have $\langle\sigma'(a),\sigma'(b)\rangle = \langle\sigma(a),\sigma(b)\rangle$ ε $F(r_A)$. By the independence of irrelevant alternatives of F this yields $\langle\sigma'(a),\sigma'(b)\rangle$ ε $F(\sigma'(\hat{r}_A))$.

By the neutrality of F we have <a,b> ε $F(\hat{r}_A)$.

Hence, <a,b> ε $\hat{F}(\hat{r}_A)$ iff <a,b> ε $F(\hat{r}_A)$.

Hence, $\hat{F}$ is a complete extension of F.

Remains to prove the uniqueness of $\hat{F}$.

Suppose H is a neutral and independent of irrelevant alternatives complete extension of F, $\hat{r}_X$ ε V_n and x,y ε X.

<x,y> ε $\hat{F}(\hat{r}_X)$, iff there is a permutation σ ε S_U and a profile r_A ε $V_n(A)$, such that $\sigma(\hat{r}|_{\{x,y\}}) = r_{A}|_{\{\sigma(x),\sigma(y)\}}$ and $\langle\sigma(x),\sigma(y)\rangle = F(r_A)$ and $\langle\sigma(x),\sigma(y)\rangle = H(r_A)$,

iff <x,y> ε $H(\hat{r}_X)$.

■

The following theorem prepares the main result of this section.

Theorem 4.1.9

Let $F : V_n(A) \rightarrow \AA$ be a welfare function on society $\Gamma = \langle A,N \rangle$, where $|A| \geq 2$ and $|N| \geq n$.

Then (4.1.9.1) up to (4.1.9.4) are equivalent.

4.1.9.1 F is symmetric and independent of irrelevant alternatives.

4.1.9.2 F is neutral and independent of irrelevant alternatives.

4.1.9.3 There exists an unique complete extension $\hat{F}$ of F, which is neutral and independent of irrelevant alternatives.

4.1.9.4 There exists an unique complete extension $\hat{F}$ of F, which is symmetric and independent of irrelevant alternatives.

Proof of theorem 4.1.9

(4.1.9.4) -> (4.1.9.1) and (4.1.9.3) -> (4.1.9.2) follow by theorem 4.1.7.

(4.1.9.2) -> (4.1.9.3) follows by theorem 4.1.8.

(4.1.9.2) <-> (4.1.9.1) and (4.1.9.4) -> (4.1.9.3) follow immediately from the following facts:

(i) every permutation is the composition of permutations of the form: $\sigma_{xy} \in S_U$, such that $\sigma_{xy}(x) = y$, $\sigma_{xy}(y) = x$ and $\sigma_{xy}(z) = z$ for $z \in U - \{x,y\}$, and

(ii) $\sigma_{xy}(r_X)|_{\{x,y\}} = \bar{v}r_X|_{\{x,y\}}$ for $X \in Æ$ and $x,y \in X$.

■

The following theorem is the main result of this section.

Theorem 4.1.10

4.1.10.1 Let $\hat{F} : V_n \rightarrow \AA$ be a complete welfare function on N, such that $|N| = n$ and V is classified as set of orderings. Then (4.1.10.1.1) and (4.1.10.1.2) are equivalent.

4.1.10.1.1 $\hat{F}$ is Pareto-optimal, independent of irrelevant alternatives and neutral.

4.1.10.1.2 $\hat{F}$ is an order morphism from V_n to $\AA$.

4.1.10.2 Let $F : V_n(A) \to \mathring{A}$ be a welfare function on $\Gamma = \langle A,N\rangle$, such that $|A| \geq 2$, $|N| = n$ and V is classified as set of orderings. Then (4.1.10.2.1) and (4.1.10.2.2) are equivalent.

4.1.10.2.1 F is Pareto-optimal, independent of irrelevant alternatives and neutral.

4.1.10.2.2 There is a unique complete extension $\hat{F}$ of F, which is an order morphism from V_n to $\mathring{A}$.

4.1.10.3 Moreover, if $|A| \geq k+1$ $w_1,w_2 \in \hat{N}^+$ (See (2.2.4.8), (2.3.5) and (2.3.8)), such that $|w_1| \leq k$, then (4.1.10.3.1) and (4.1.10.3.2) are equivalent.

4.1.10.3.1 $\hat{F}(V_n(A)) \subseteq \{R_A : R_A$ is $\langle w_1,w_2\rangle$-transitive$\}$.

4.1.10.3.2 $\hat{F}(V_n) \subseteq \{R_X \in \mathring{A} : R_A$ is $\langle w_1,w_2\rangle$-transitive$\}$.

<u>Proof of theorem 4.1.10</u>

(4.1.10.1.2) $\to$ (4.1.10.1.1) Suppose $\hat{F}$ is an order morphism form V_n to $\mathring{A}$. We have to prove that $\hat{F}$ is Pareto-optimal, independent of irrelevant alternatives and neutral. The fact that $\hat{F}$ is neutral and independent of irrelevant alternatives follows from the definition of an order morphism.

To prove that $\hat{F}$ is Pareto-optimal it suffices to prove that $\langle a,b\rangle \in \bar{a}\hat{F}(r_A)$, if $\langle a,b\rangle \in \bar{a}R^i_A$ for all $i \in N$, all $r_A = \langle R^1_A,\ldots,R^n_A\rangle \in V_n$ and all $a,b \in A$.

Let $r_A \in V_n$, with $a,b \in A$ and $\langle a,b\rangle \in \bar{a}R^i_A$ for all $i \in N$.

Then $r_{A|_{\{a\}}} \gg r_{A|_{\{b\}}} \,\hat{=}\, r_{A|_{\{a,b\}}}$.

Hence, $F(r_{A|_{\{a,b\}}}) = F(r_{A|_{\{a\}}} \gg r_{A|_{\{b\}}})$

$$= \hat{F}(r_A)|_{\{a\}} \gg \hat{F}(r_A)|_{\{b\}}.$$

So $\langle a,b\rangle \in \bar{a}\hat{F}(r_A)$.

(4.1.10.1.1) $\to$ (4.1.10.1.2) Suppose $\hat{F}$ is an independent of irrelevant alternatives, neutral and Pareto-optimal complete welfare function.

Clearly $\hat{F}$ is a complete extension of $\hat{F}|_{V_n(A)}$, where $|A| \geq 2$.

By (4.1.9) it suffices to prove that

$\hat{F}(r_A \gg r'_B) = \hat{F}(r_A) \gg \hat{F}(r'_A)$ and

$\hat{F}(Sub(r_X,x,r'_Y)) = Sub(\hat{F}(r_X),x,\hat{F}(r'_Y))$ for suitable r_A,r'_B,r_X,r'_Y and x.

Let $r_A,r'_B \in V_n$, such that $A \cap B = \varphi$.

It is sufficient to prove that $(A \times B)_{A \cup B} \subseteq \bar{a}\hat{F}(r_A \gg r'_B)$.

Take $a \in A$ and $b \in B$. Since $\hat{F}$ is Pareto-optimal we have

$\bar{a}(R^1_A|_{\{a\}} \gg \hat{R}'^1_B|_{\{b\}}) \subseteq \bar{a}\hat{F}(r_A \gg r'_B)|_{\{a,b\}}$.

So $\langle a,b\rangle \in \bar{a}\hat{F}(r_A \gg r'_B)$.

Let $r_X,r'_Y \in V_n$ and $a \in X$, such that $X \cap Y = \varphi$ and $\bar{v}r'_Y = r'_Y$.

It issufficient to prove that for all $x,y \in X \cup Y-\{a\}$

$Sub(\hat{F}(r_X),a,\hat{F}(r'_Y))|_{\{x,y\}} = \hat{F}(Sub(r_X,a,r'_Y))|_{\{x,y\}}$.

We distinguish four cases.

Case 1 $x,y \in X - \{a\}$ $Sub(\hat{F}(r_X),a,\hat{F}(r'_Y))|_{\{x,y\}} = \hat{F}(r_X)|_{\{x,y\}}$ and

$\hat{F}(Sub(r_X,a,r'_Y))|_{\{x,y\}} = \hat{F}(Sub(r_X,a,r'_Y)|_{\{x,y\}}) = F(r_X|_{\{x,y\}})$

$= \hat{F}(r_X)|_{\{x,y\}}$. Hence, this case is done.

Case 2 $x,y \in Y$. It is similar to case 1.

Case 3 $x \in X - \{a\}$ and $y \in Y$ Let $\sigma_{ay} \in S_U$ be with $\sigma_{ay}(a) = y$, $\sigma_{ay}(y) = a$ and $\sigma_{ay}(z) = z$, for all $z \in U - \{y,a\}$.

Then $Sub(r_X,a,r'_Y)|_{\{x,y\}} = \sigma_{ay}(r_X)|_{\{x,y\}}$.

Hence, $\hat{F}(Sub(r_X,a,r'_Y))|_{\{x,y\}} = \hat{F}(\sigma_{ay}r_X)|_{\{x,y\}}$

$= \sigma_{ay}(\hat{F}(r_X))|_{\{x,y\}} = \sigma_{ay}(\hat{F}(r_X))|_{\{x,y\}} = Sub(\hat{F}(r_X),a,\hat{F}(r'_Y))$.

Case 4 $x \in Y$ and $y \in X - \{a\}$.It is similar to case 3.

(4.1.10.2.1) $\leftrightarrow$ (4.1.10.2.2) follows evidently.

(4.1.10.3.2) $\rightarrow$ (4.1.10.3.1) is trivial.

(4.1.10.3.1) $\rightarrow$ (4.1.10.3.2)

Suppose $R_X \in \hat{F}(V_n)$ is not $\langle w_1,w_2\rangle$-transitive. Then there is a path of length k+1 and type w_1 which cannot be cut short by a path of type $w_3 \in \hat{N}^+$ embedded in w_2. Let $\langle x_0,\dots x_k\rangle$ be that path. Take $\sigma \in S_U$ with $\sigma(\{x_0,\dots x_k\}) \subseteq A$.

Then $(\sigma R_X)|_A \in \hat{F}(V_n(A))$ and $\langle\sigma(x_0),\dots\sigma(x_k)\rangle$ is a path along $(\sigma R_X)|_A$ of type w_1 which cannot be cut short by a path of type w_3 embedded in w_2. Hence, $(\sigma R_X)|_A$ is not $\langle w_1,w_2\rangle$-transitive. ■

If we are willing to accept that an order homomorphism is a transformation between sets of profiles or relations, (which preserves the essential aspects of orderings,) then by theorem 4.1.10 it follows that the constraints Pareto-optimality, neutrality and independence of irrelevant alternatives, (constraints which stem from formalizations of "social" conditions,) are necessary and sufficient to preserve the orderings aspects of profiles. Moreover, by the same theorem 4.1.10 it follows that a Pareto-optimal, independent of irrelevant alternatives and neutral welfare function corresponds with an order morphism in an unique way.

This last fact applies that the study of welfare functions can for a great deal be done in the "framework" of order morphism, since the three conditions are frequently imposed (sometimes implicit) on welfare functions. This study will be done in section 4.3. Furthermore, it is mentioned that, although these three conditions are very strong and therefore often critized, they imply at least theoretically desirable properties as is shown in theorem 4.1.10 and corollary 4.1.11.

Corollary 4.1.11

Let $F : V_n(A) \rightarrow \mathring{A}$ be a welfare function on $\Gamma = \langle A,N \rangle$, such that $|A| \geq 2$ and $|N| = n$. Furthermore, let F be Pareto-optimal, independent of irrelevant alternatives and neutral.

Then there is an unique complete extension $\hat{F}$ of F, such that $\hat{F}$ is an order morphism, $\hat{F}(V_n)$ can be classified as a set of orderings and $\hat{F}(V_n(A)) = \hat{F}(V_n)(A)$.

Proof of corollary 4.1.11

follows from the foregoing theorems.

■

§ 4.2 Revealed Rationality and Choice Correspondences

In Social Choice Theory, decision rules are studied by both welfare functions and choice correspondences. In relation to choice correspondences one sometimes treated welfare functions as the underlying "rational behaviour" of the society by which its choice is determined. To be more precize: One often supposes that collective choices of a group (the outcomes of a choice correspondence at a specific profile) are based on collective orderings (the outcomes of a welfare function).

Moreover, one often supposes that these choices correspond with the best elements according to a collective ordering. (See e.g. Gibbard [1973], Satterthwaite [1975], Kelly [1978], Arrow [1978], Kalai & Muller [1977], Richter [1966], Plott [1976] and Ritz [1985]).

This idea of "rational behaviour" will be generalized to the notion of reconstructability by orderings. This is done by means of a recently developed notion of "rationality" in Revealed Preference theory (See Ruys & Storcken [1988]). It appears, that a choice correspondence can be reconstructed by orderings, iff it is unanimity respecting, neutral, independent of irrelevant alternatives and uniform extendable from the binary choices. These properties are often implicitly or explicitly imposed on the choice correspondences. These results make it possible to study choice correspondences by studying order morphisms.

Let us develop these results in a formal way. In the first chapter the notion of a choice correspondence C on a society $\Gamma = \langle A,N \rangle$ has been defined as a function from a set of profiles on A to the powerset of A, with the constraint that the empty set is not in the range of C. Let V be a set of relations which is classified as set of orderings. In the terminology of chapter 2 and section 4.1, C becomes a function from $V_n(A)$ to 2^A, such that for all $r_A \in V_n(A)$, $C(r_A) \neq \varphi$, and $|N| = n$. Although a choice correspondence carries a lot of essential information, this model will be extended here to a richer one, in order to make the reconstructability property more approriate to collective decision rules. This extended form has been introduced earlier in literature (See e.g. Kelly [1978] and Fishburn [1973]).

Definition 4.2.1 Collective choice rule

Suppose $\Gamma = \langle A,N \rangle$ is a society, with $|N| = n$, and V is a set of relations, such that V is classified as a set of orderings. A function K from $(2^A - \{\varphi\}) \times V_n(A)$ to 2^A, such that $\varphi \neq K(X,r_A) \subseteq X$, for all $X \in 2^A - \{\varphi\}$ and all $r_A \in V_n(A)$, is a <u>collective choice rule on</u> Γ from $(2^A - \{\varphi\}) \times V_n(A)$ to $2^A - \{\varphi\}$.

■

In chapter 1 we have introduced some conditions for choice correspondences. Now we add some new conditions and several conditions for collective choice rules.

Definition 4.2.2

Let $\Gamma = \langle A,N \rangle$ be a society, such that $|A| = p$ and $|N| = n$. Furthermore, let V be a classified set of orderings, C a choice correspondence from $V_n(A)$ to 2^A on Γ and K a collective choice rule on Γ from $(2^A - \{\varphi\}) \times V_n(A)$ to 2^A.

4.2.2.1 K is <u>neutral</u>, iff $\sigma K(X,r_A) = K(\sigma(X),\sigma r_A)$, for all $\sigma \in S_A$ and all $(X,r_A) \in (2^A - \{\varphi\}) \times V_n(A)$.

4.2.2.2 C is <u>independent of irrelavant alternatives</u>, iff for all $r_B \gg r_D$, $\hat{r}_B \gg \tilde{r}_D$ in $V_n(A)$: $C(r_B \gg r_D) = C(\hat{r}_B \gg \tilde{r}_D)$.

4.2.2.3 K is <u>independent of irrelevant alternatives</u>, iff for all r_A, $\hat{r}_A \in V_n(A)$ and $\varphi \neq X \subseteq A$: if $r_{A|_X} = \hat{r}_{A|_X}$, then $K(X,r_A) = K(X,\hat{r}_A)$.

4.2.2.4 K is <u>unanimity respecting</u>, iff $K(X, r_B \gg r_D) \subseteq B$, for all $r_B \gg r_D \in V_n(A)$ and all $\varphi \neq X \subseteq A$, with $B \subseteq X$.

■

Let us comment on definition 4.2.2. Neutrality is a standard definition. The independence of irrelevant alternatives condition is well-known for welfare functions and collective choice rules (See Kelly [1973]). As defined here it is new for choice correspondences. This condition as defined in (4.2.2.2) resembles the equally named condition for welfare functions at the interpretative level, that is the collective choice only depends on elements among which society has to determine its choice. This explaines its name. The respect of unanimity can be interpreted as follows: if the individuals strictly prefer all elements in B above all elements in D, that is the profile is of the form $r_B \gg r_D$, then the collective choice of society is in B. Clearly, in that case all individuals order unanimously all elements in B above all elements in D and the collective decision respects this unanimity. This explaines the name.

Next we prove that every neutral, independent of irrelevant alternatives and unanimity respecting choice correspondence can be extended to a collective choice rule.

<u>Theorem 4.2.3</u> <u>Extension of choice correspondences</u>

Let $\Gamma = \langle A,N \rangle$ be a society, with $|N| = n$, $V \subseteq \mathring{A}$ being classified as a set of orderings and C a neutral independent of irrelevant alternatives and Pareto-optimal choice correspondence from $V_n(A)$ to 2^A on Γ. Then there exists an unique collective choice rule K from $(2^A - \{\varphi\}) \times V_n(A)$ to 2^A on Γ, such that:

4.2.3.1 K is neutral, independent of irrelevant alternatives and unanimity respecting,

4.2.3.2 for all $r_A \in V_n(A)$: $K(A,r_A) = C(r_A)$, and

4.2.3.3 for all $X,Y \in 2^A - \{\varphi\}$, with $X \subseteq Y$, and all $r_A \in V_n(A)$: $K(X, r_{A|_X} \gg r_{A|_{A-X}}) = K(Y, r_{A|_X} \gg r_{A|_{A-X}})$.

<u>Proof of theorem 4.2.3</u>

Suppose Γ, V and C as above.

Define $K(X,r_A) := C(r_{A|_X} \gg r_{A|_{A-X}})$.

Clearly K is a collective choice rule from $(2^A - \{\varphi\}) \times V_n(A)$ to 2^A on Γ, which satisfies (4.2.3.2).

Since $\sigma C(r_{A|_X} \gg r_{A|_{A-X}}) = C(\sigma(r_{A|_X} \gg r_{A|_{A-X}}))$

$= C(\sigma r_{A|_{\sigma(X)}} \gg \sigma r_{A|_{\sigma(A-X)}})$ for all

permutations $\sigma \in S_A$ it follows that K is neutral. The independence of irrelevant alternatives of K follows from the independence of irrelevant alternatives of C.

Suppose $\varphi \neq B \subseteq X \subseteq A$ and $r_B \gg \bar{r}_D \in V_n(A)$.

$K(X,r_B \gg \bar{r}_D) = C((r_B \gg \bar{r}_D)|_X \gg (r_B \gg \bar{r}_D)|_{A-X})$

$= C(r_B \gg \bar{r}_{D|_X} \gg \bar{r}_{D|_{D-X}}) \subseteq B.$

Hence, (4.2.3.1) holds for K.

(4.2.3.3) suppose $\varphi \neq X \subseteq Y \subseteq A$ and $r_A \in V_n(A)$.

It is sufficient to prove that

$C(r_{A|_X} \gg r_{A|_{A-X}}) = C((r_{A|_X} \gg r_{A|_{A-X}})|_Y \gg (r_{A|_X} \gg r_{A|_{A-X}})|_{A-Y})$.

Since C is independent of irrelevant alternatives this is obvious.

Remains to prove the uniqueness of K.

Suppose K' is a collective choice rule on Γ from $(2^A - \{\varphi\}) \times V_n(A)$ to 2^A which satisfies (4.2.3.1), (4.2.3.2) and (4.2.3.3).

Let $X \subseteq 2^A - \{\varphi\}$ and $r_A \in V_n(A)$. It is sufficient to prove that $K'(X,r_A) = K(X,r_A)$.

$$
\begin{aligned}
K'(X,r_A) &= K'(X,r_{A|_X} \gg r_{A|_{A-X}}) && \text{(by the independence)}\\
&= K'(A,r_{A|_X} \gg r_{A|_{A-X}}) && \text{(by (4.2.3.3))}\\
&= C(r_{A|_X} \gg r_{A|_{A-X}}) && \text{(by (4.2.3.2))}\\
&= K(X,r_A) && \text{(by definition of K).}
\end{aligned}
$$

■

Theorem 4.2.3 states that a Pareto-optimal, neutral and independence of irrelevant alternatives choice correspondence determines a unique type of collective choice rules. By this property of choice correspondences it is sufficient to study only collective choice rules.

This study is continued by a formulation of the notion of reconstructability by orderings borrowed from the theory of Revealed Preference. This theory is usually concerned with the revelation of a preference relation of a decision maker. In our approach this decision maker is the society. The revealed preference is just as in Revealed Preference Theory, a model variable. By this variable and some others the choice behaviour of the decision maker, the society, can be reconstructed within the model. That is, all the independent variables of the model can have a value which lead to an unique value of all model variables (and so for the model) such that the choice behaviour coincides with this value of the model. The knowledge of this preference relation is not sufficient for reconstructing choice behaviour of that decision maker. One also has to know the procedure used by the society to arrive at a decision, such as choosing a best ordered element, a maximal element, a conflict minimizing element, or any other decision criterion. This procedure should be revealed from the choice behaviour, together with the preference relation (For more information see Ruys & Storcken [1988]). So such a procedure is a variable of the model. These ideas are formulated as follows.

Definition 4.2.4 Decision procedure

Let $W \subseteq \mathring{A}$ be a classified set of orderings. A decision procedure H on W is a function from $\alpha(W) := \{\langle X,R_Y\rangle : X \subseteq Y,\ X \neq \varphi \text{ and } R_Y \in W\}$ to Æ, such that

4.2.4.1 $H(X,R_Y) \subseteq X$, for all $\langle X,R_Y\rangle \in \alpha(W)$,

4.2.4.2 $\sigma H(X,R_Y) = H(\sigma(X),\sigma R_Y)$, for all $\langle X,R_Y\rangle \in \alpha(W)$ and all $\sigma \in S_U$ (neutrality),

4.2.4.3 $H(X,R_Y) = H(X,R'_Y)$, for all $\langle X,R_Y\rangle$, $\langle X,R'_Y\rangle \in \alpha(W)$, with $R_Y|_X = R'_Y|_X$ (independence of irrelevant alternatives),

4.2.4.4 $H(X,R_B \gg R'_D) \subseteq X$, for all $\langle X,R_B \gg R'_D\rangle \in \alpha(W)$, with $B \subseteq X$ (preference consistent).

∎

A decision procedure is an extended neutral, independent of irrelevant alternatives and unanimity respecting collective choice rule for a society with one single individual.

Now the notion of reconstructability by orderings is defined.

Definition 4.2.5 Reconstructability by orderings

Suppose $\Gamma = \langle A,N\rangle$ is a society, with $|N| = n$, $V \subseteq \mathring{A}$ is a classified set of orderings and K is a collective choice rule on Γ from $(2^A - \{\varphi\}) \times V_n(A)$ to 2^A.

K is reconstructable by orderings, iff there exist an order morphism $\hat{F}$ from V_n to $\mathring{A}$ and a decision procedure H from $\alpha(\Phi(\hat{F}(V_n(A))))$ to Æ, such that for all $\langle X,r_A\rangle \in (2^A - \{\varphi\}) \times V_n(A) : K(X,r_A) = H(X,\hat{F}(r_A))$.

In that case we say K is reconstructed by $\hat{F}$ and H.

($\Phi(W)$ is the smallest possible set containing W, which can be classified as a set of orderings (See also (2.4.4))).

∎

A collective choice rule is reconstructed by orderings, iff at every profile r_A and agenda $X \subseteq A$, it is possible to describe the choice of K by the choice of a decision procedure H at that agenda X and by $\hat{F}(r_A)$, the outcome of order morphism $\hat{F}$ under r_A. The following theorem characterizes by orderings reconstructable collective choice correpondences. The results of §4.1 the correpondence between those correspondences and special welfare functions become apparent.

In Kalai & Muller [1977], and Ritz [1985] also correspondences between collective choice rules and welfare functions are proved. However, because of the differences between (social) conditions imposed on those functions, these results and those of the next theorem are not easily comparable.

Theorem 4.2.6

Suppose $\Gamma = \langle A,N \rangle$ is a society, with $|N| = n$ and $|A| = p \geq 2$, $V \subseteq \mathring{A}$ is a classified set of orderings and K is a collective choice rule from $(2^A - \{\varphi\}) \times V_n(A)$ to $2^A - \{\varphi\}$. Then (4.2.6.1) and (4.2.6.2) are equivalent.

4.2.6.1 K is reconstructed by orderings.

4.2.6.2 K is neutral, independent of irrelevant alternatives, unanimity respecting and uniform extended from the binary choices, i.e., for all $X \subseteq A$, with $|X| \geq 3$, and all $r_A, r_A' \in V_n(A)$: if for all $Y \subset X$, with $Y \neq \varphi$, $K(Y,r_A) = K(Y,r_A')$, then $K(X,r_A) = K(X,r_A')$.

4.2.6.3 Furthermore, if K is reconstructed by orderings, then there are unique $\hat{F}$ and H, such that $\hat{F}(V_n)$ is a set of complete relations and K is reconstructed by $\hat{F}$ and H.

Proof of theorem 4.2.6

(4.2.6.1) -> (4.2.6.2). Suppose (4.2.6.1). Then by definition there is a decision procedure H from $\alpha(\Phi(\hat{F}(V_n(A))))$ to Æ and an order morphism $\hat{F}$ from V_n to $\mathring{A}$, such that $K(X,r_A) = H(X,\hat{F}(r_A'))$ for all $X \subseteq A$, with $X \neq \varphi$, and all $r_A \in V_n(A)$.

(neutrality) Take $\sigma \in S_A$, $\varphi \neq X \subseteq A$ and $r_A \in V_n(A)$.

$$\begin{aligned}\sigma K(X,r_A) &= \sigma H(X,\hat{F}(r_A))\\ &= H(\sigma(X),\sigma\hat{F}(r_A))\\ &= H(\sigma(X),\hat{F}(\sigma r_A')) = K(\sigma(X),\sigma r_A).\end{aligned}$$

(independence of irrelevant alternatives) Take $r_A, r'_A \in V_n(A)$, $\varphi \neq X \subseteq A$, with $r_A|_X = r'_A|_X$.

Then $\hat{F}(r_A)|_X = \hat{F}(r'_A)|_X$ and $H(X,\hat{F}(r_A)) = H(X,\hat{F}(r'_A))$

$$K(X,r_A) = H(X,F(r_A))$$
$$= H(X,F(r'_A)) = K(X,r'_A).$$

(unanimity respect) Take $\hat{r}_B \gg r'_D \in V_n(A)$ and $B \subseteq X \subseteq A$,

$$K(X,\ r_B \gg r'_D) = H(X,F(r_B \gg r'_D))$$
$$= H(X,\hat{F}(r_B) \gg \hat{F}(r'_D))$$
$$\subseteq B.$$

(uniform extension) Take $X \subseteq A$, $|X| \geq 3$ and $r_A, r'_A \in V_n(A)$, such that for all $Y \subset X : K(Y,r_A) = K(Y,r'_A)$.

Then for all $x,y \in X : K(\{x,y\},r_A) = K(x,y\},r'_A)$.

Now by the neutrality, the preference consistent and the independence of irrelevant alternatives of H it follows:

(i) not $x > y\ \hat{F}(r_A)$ iff $x \in K(\{x,y\},r_A)$,

(ii) not $x > y\ \hat{F}(r'_A)$ iff $x \in K(\{x,y\},r'_A)$, and

(iii) $\hat{F}(r_A)$ is either complete or antisymmetric.

Hence, $\hat{F}(r_A)|_X = \hat{F}(r'_A)|_X$ and therefore

$$K(X,r_A) = H(X,\hat{F}(r_A))$$
$$= H(X,\hat{F}(r'_A)) = K(X,r'_A).$$

(4.2.6.2) → (4.2.6.1) Suppose (4.2.6.2).

First we define a welfare function F from $V_n(A)$ to Å for all $x,y \in A$ and $r_A \in V_n(A)$ as follows:

$x \geq y : F(r_A)$ iff $x \in K(\{x,y\},r_A)$.

Evidently F is a welfare function.

By the independence of irrelevant alternatives, the neutrality and the unanimity respect of K it follows straightforward that F is independent of irrelevant alternatives, neutral and Pareto-optimal.

Hence, there is an unique complete extension $\hat{F}$ of F from V_n to $F(V_n)$, which is an order morphism by (4.1.10).

Note that the correspondence of $\hat{F}$ and K is unique. (4.2.6.4)

By (2.4.3) and (2.4.4) $\Phi(F(V_n(A))) = \Sigma_4\Sigma_6\Sigma_3\Sigma_1\Sigma_2(\hat{F}(V_n(A)))$.

From the proof of (4.1.9) it follows that F is symmetric.

So, since $\{\bar{v}r_A : \hat{r}_A \in V_n(A)\} = V_n(A)$, it follows that $\Sigma_2(\hat{F}(V_n(A))) = F(V_n(A))$ and consequently we have $\Phi(F(V_n(A))) = \Sigma_4\Sigma_6\Sigma_3\Sigma_1(F(V_n(A)))$.

Notice that $\hat{F}(V_n(A))$ is a set of strongly complete relations. Therefore $\Phi(\hat{F}(V_n(A)))$ is a set of strongly complete relations.

For all $x,y \in X$ and $R_X \in \AA$ define the equivalence relation $E(R_X)$ as follows: $x \sim y : E(R_X)$, iff

$$x \geq z : R_X, \text{ iff } y \geq z : R_X \text{ for all } z \in X.$$

Now for an arbitrary $\langle Y,R_X\rangle \in \alpha(\Phi(\hat{F}(V_n(A))))$ define

$H(\langle Y,R_X\rangle) := \cup \{B_i : d_i \in \sigma(K(\sigma^{-1}(D),r_A))\}$, where

(i) $R_X|_Y = R_Z \gg R_X|_{Y-Z}$ and R_Z is irreducible,

(ii) $B_1,B_2,\ldots.B_t$ are the equivalence classes of $E(R_Z)$

$D = \{d_1,d_2,\ldots d_t\}$ and $d_i \in B_i$ for all $i \in \{1,..t\}$, and

(iii) $R_Z|_D = \sigma(\hat{F}(r_A)|_{\sigma^{-1}(D)})$ for a $r_A \in V_n(A)$ and a $\sigma \in S_U$.

Note that R_Z, $E(R_Z)$, σ and r_A exist.

Moreover, R_Z and $E(R_Z)$ are unique.

(well-definedness of $H(\langle Y,R_X\rangle)$) Suppose $D' = \{d_1',d_2',\ldots d_t'\}$ and $d_i' \in B_i$ for all $i \in \{1,..t\}$, $R_Z|_{D'} = \sigma(\hat{F}(r_A')|_{\sigma'^{-1}(D)})$ where $r_A' \in V_n(A)$ and a $\sigma' \in S_U$.

It is sufficient to prove that

$\sigma'(K(\sigma'^{-1}(D'),r_A')) = \sigma(K(\sigma^{-1}(D),r_A'))$.

Let $\tau \in S_U$ be such that $\tau(d_i') = d_i$ and $\tau(d_i) = d_i'$ for all $i \in \{1,\ldots,t\}$ and $\tau(x) = x$ for all $x \in U - (D \cup D')$.

Now $\tau(R_Z|_{D'}) = R_Z|_D$ and $\tau(R_Z|_D) = R_Z|_{D'}$.

Hence, $\sigma(\hat{F}(r_A)|_{\sigma^{-1}(D)}) = R_Z|_D = \tau\sigma'(\hat{F}(r_A')|_{\sigma'^{-1}(D')})$.

So $\hat{F}(r_A)|_{\sigma^{-1}(D)} = \sigma^{-1}\tau\sigma'(\hat{F}(r_A')|_{\sigma'^{-1}\tau^{-1}(D)})$.

Now $\sigma^{-1}\tau\sigma'(A) = A$ so

$\hat{F}(r_A)|_{\sigma^{-1}(D)} = \hat{F}(\sigma^{-1}\tau\sigma'(r_A'))|_{\sigma^{-1}(D)}$.

Hence, for all $x,y \in \sigma^{-1}(D)$:

$K(\{x,y\},r_A) = K(\{x,y\},\sigma^{-1}\tau\sigma'r_A'))$.

By the uniform extension of the binary choices property of K and a simple induction reasoning it follows:

$$\begin{aligned} K(\sigma^{-1}(D),r_A) &= K(\sigma^{-1}(D),\sigma^{-1}\tau\sigma'r_A')) \\ &= \sigma^{-1}\tau\sigma'K(\sigma^{-1}\tau\sigma'(D),r_A') \\ &= \sigma'^{-1}\tau\sigma'K(\sigma'^{-1}(D'),r_A'). \end{aligned}$$

(neutrality, independentness and preference consistency) follow straightforwardly.

Note that the correspondence of H and $\hat{F}$ and K is unique.

(4.2.6.5)

(4.2.6.3) Follows immediately from the equivalence and (4.2.6.3) and (4.2.6.5).

■

It may be clear to the reader that the definition of $F(r_A)$ in theorem 4.2.6 is standard in social choice theory. Often it is referred to as base relation (See Kelly [1978]).

Characterizations of, by orderings reconstructable, collective choice rules are found in literature only for the case where the decision procedure, H, chooses the best elements of $\hat{F}(r_A)$, i.e., best$(\hat{F}(r_A)|_X) = H(X,\hat{F}(r_A))$, (See Kelly [1978], Bordes [1979], Blair & Bordes & Kelly & Suzumura [1976] and Satterthwaite [1975]). Since in our approach H is not specified in such a specific way theorem 4.2.6 is a small extension to the results in literature.

This section concludes by showing that reconstructability by orderings, independence of irrelevant alternatives, neutrality, unanimity respect and uniform extendability from the binary choices are implied by well-known social constraints for collective choice rules and choice correspondences. Furthermore, it is shown that impossibility results for welfare functions can be translated to impossibility results for choice correspondences.

In chapter 1 the strong positive association has been introduced for choice correspondences. This condition is frequently imposed on correspondences (See e.g. Muller & Satterthwaite [1977], Moulin [1983] and Peleg [1984]) along with another condition for choice correspondences, namely, single-valuedness, i.e., the outcomes of a choice correspondence are singletons. The former is implied by the well-known non-manipulability condition. (For this result see e.g. Muller & Satterthwaite [1977]). Now the two conditions non-manipulability and single-valuedness appear in many impossibility theorems starting with Gibbard [1973] and Satterthwaite [1975]. Both have proven firstly, independently from each other, an impossibility theorem in the sense of Arrow for choice correspondences.

Now we will prove that a single-valued, Pareto-optimal and strongly positively associated choice correspondence C has a by orderings reconstructable extension K, i.e., a collective choice rule as defined in (4.2.3).

<u>Theorem 4.2.7</u>

Suppose $\Gamma = \langle A,N \rangle$, with $|N| = n$ and $|A| = p \geq 3$, is a society, C is a Pareto-optimal, single-valued and strongly positively associated choice correspondence from $L_n(A)$ to $(2^A - \{\varphi\})$.

Then $K : (2^A - \{\varphi\}) \times L_n(A) \rightarrow 2^A$, defined by

$K(X,r_A) := C(r_{A|_X} \gg r_{A|_{A-X}})$ for all $X \in 2^A - \{\varphi\}$ and all $r_A \in L_n(A)$, can be reconstructed by orderings.

<u>Proof of theorem 4.2.7</u>

For all $x,y \in A$ and $r_A \in V_n(A)$ define

$r_A^{xy} := r_{A|_{\{x,y\}}} \gg r_{A|_{A-\{x,y\}}}$.

By theorem 4.2.3 and 4.2.6 it is sufficient to prove that C is independent of irrelevant alternatives, neutral and K is uniform extended from the binaire choices.

First we prove an assertion:

<u>Assertion 4.2.7.1</u> For all $r_A \in V_n(A)$:

$\{x\} = C(r_A)$ iff for all $y \in A$: $\{x\} = C(r_A^{xy})$.

<u>Proof of assertion 4.2.7.1</u>

(only if) follows immediately from the strong positive association.

(if) Suppose $\{x\} = C(r_A^{xy})$ for all $y \in A$. Then from the (only if) $C(r_A) \neq \{z\}$ for every $z \in A - \{x\}$. So $C(r_A) = \{x\}$.

From this assertion and the Pareto-optimality of C it follows straightforwardly that C is independent of irrelevant alternatives.

Next we prove that K is uniformly extended from the binary choices.

Suppose $K(\{x,y\},r_A) = K(\{x,y\},r'_A)$ for all $x,y \in D \subseteq A$.

It suffices to prove that

$C(r_{A|_D} \gg r_{A|_{A-D}}) = C(r'_{A|_D} \gg r'_{A|_{A-D}})$.

From the assumption it follows: $C(r_A^{xy}) = C(r_A'^{xy})$ for all $x,y \in D$.

Hence, from the independence of irrelevant alternatives of C it follows $C((r_{A|_D} \gg r_{A|_{A-D}})^{xy}) = C((r'_{A|_D} \gg r'_{A|_{A-D}})^{xy})$ for all $x,y \in D$.

By the Pareto-optimality of C and assertion 4.2.7.1 it follows $C(r_{A|_D} \gg r_{A|_{A-D}}) = C(r'_{A|_D} \gg r'_{A|_{A-D}})$.

So K is uniform extended from the binary choices.

To prove that C is neutral it is sufficient to prove that if $S \subseteq N$ decides x against y, then S decides x against z and S decides z against y for all $z \in A - \{x,y\}$.

Here S decides x against y, iff

$S = \{i \in N : x > y : R_A^i\}$ and $\{x\} = C(r_A^{xy})$.

Suppose S decides x against y and let $z \in A - \{x,y\}$.

Take r_A : $x > y > z > t : R_A^i$, for all $i \in S$ and all $t \in A - \{x,y,z\}$, and

$y > z > x > t : R_A^i$, for all $i \in N\text{-}S$ and all $t \in A - \{x,y,z\}$.

$C(r_A) \neq \{y\}$, since C is strongly positively associated and $C(r_A^{xy}) = \{x\}$.

$C(r_A) \neq \{t\}$, for all $t \in A - \{x,y\}$, since C is Pareto-optimal.

Hence, $\{x\} = C(r_A)$ and by the strong positive association of C we have $C(r_A^{xy}) = \{x\}$.

Hence, S decides x against z.

Similarly it follows that S decides z against y.

■

It is clear that the last part of the proof of (4.2.7) is inspired by Arrow's proof of his impossibility theorem (See Arrow [1978]). Now we prove that C is dictatorial, a result first proved by Muller & Satterhwaite [1977].

Theorem 4.2.8

Suppose $\Gamma = \langle A,N \rangle$, with $|N| = n$ and $|A| = p \geq 3$, is a society and C is a Pareto-optimal, single-valued and strongly positively associated choice correspondence from $L_n(A)$ to $(2^A - \{\varphi\})$.

Then C is dictatorial, i.e., there is an individual $i \in N$ with $C(r_A) = \text{best}(R_A^i) := \{x \in A : x \geq y : R_A^i \text{ for all } y \in A\}$.

Proof of theorem 4.2.8

Let $K : (2^A - \{\varphi\}) \times L_n(A) \rightarrow (2^A - \{\varphi\})$ be defined by $K(X,r_A) := C(r_{A|_X} \gg r_{A|_{A-X}})$ for all $X \in 2^A - \{\varphi\}$ and

$r_A \in \hat{L}_n(A)$. By theorem 4.2.6 and 4.2.7 K is reconstructed by say F and H, where $\hat{F}$ is the complete extension of F.

Note that $x > y : F(r_A)$ iff $\{x\} = C(r_A^{xy})$. (4.2.8.1)

So $F(L_n(A)) \subseteq T(A)$.

We prove that $F(r_A) \in L(A)$, Let $x > y : F(r_A)$ and $y > z : F(r_A)$.

It suffices to prove $x > z : F(r_A)$.

Suppose $z > x : F(r_A)$.

Take $r_A' := r_{A|_{\{x,y,z\}}} \gg r_{A|_{A-\{x,y,z\}}}$.

Now by the positive association of C it follows that $C(r_A') \cap \{x,y,z\} = \varphi$. So C is not Pareto-optimal.

This cannot be the case, therefore $x > z : F(r_A)$.

Hence, by (4.1.10.3) $\hat{F}(L_n(U)) = L(U)$.

Now we use some knowledge developed in §4.3. So $\hat{F}$ is dictatorial and consequently F is dictatorial. It follows that there is an $i \in N$ such that $R_A^i = F(r_A)$ for all $r_A \in L_n(U)$. By (4.2.8.1) and (4.2.7.1), it follows that i is a dictator of C.

■

In this section it is shown that special choice correpondences and collective choice rules can be translated to order morphisms. This translation is unique and makes it possible to derive impossibility theorems for choice correspondences from impossibility theorems for welfare functions.

§ 4.3 Impossibility Theorems Based on the Independence of Irrelevant Alternatives.

This section deals with the Arrow-Paradox. The theorems discussed here have the following nature: Several conditions imposed on welfare functions jointly yield a contradiction. Or stated otherwise, welfare functions cannot satisfy jointly those conditions, or there do not exist welfare functions which satisfy all those conditions simultaneously. The set of conditions varies from theorem to theorem, but in all those sets each of the following conditions or types of conditions are present (maybe along with others not stated here):

A. the independence of irrelevant alternatives condition,

B. conditions which state that the outcome of a welfare function is not dictated by a (small) set of individuals,

C. conditions, which guarantee that the domain of the welfare contains enough profiles, such that the other conditions are meaningful and interfere with each other, and

D. conditions on the range of the welfare functions. These conditions require that the image of a welfare function is in some specific way related to its domain, e.g., transitivity or acyclicity.

In this section these conditions are studied in terms of order morphisms. It is pointed out how these contradictions are brought about and this knowledge is used to deduce stronger results than those in literature. Furthermore, by virtue of the classification system for orderings we are able to formulate theorems, which are completely new.

We start of with order morphisms on classified sets of profiles. Hence, by theorem 4.1.10 we are studying Pareto-optimal, neutral and independent of irrelevant alternatives welfare functions on a classified set of profiles restricted to a domain A (a set of alternatives). The first property, Pareto-optimality, is in all those impossibility theorems at least implicit, the second is nearly always implicitly proven and the third is standardly imposed.

Although in many impossibility theorems the domain is a restriction of a classified set of profiles, this is not always the case, (see Barthélémy [1983] and Storcken [1984]). Those restrictions on the domain can take place, because of the fact that the rest of the set of profiles restricted to A is not used in the proof of the impossibility theorems. Here we simplify our model and do consider all unused profiles as well. Furthermore, although impossibility theorems on restricted domains give an indication on the frequency by which they may occur in general and are perhaps weaker in a logical sense, these restrictions are not essential. We have some results of the following kind. If there exists a "nice" welfare function on the restricted domains as in Storcken [1984], then there exists an extension to $L_n(A)$ of this welfare function.

Recall that Pareto-optimality, independence of irrelevant alternatives and neutrality can be formulated in terms of decisiveness, whenever the domain of the welfare function is a subset of T(U), the set of tournaments. This is demonstrated in section 1.6. This decisiveness language enables comparisons of various conditions imposed on a welfare function. Hence, it is not surprising that one of the conditions is of a non-dictatorial type. Furthermore, this language enables several non-cooperative game theoretical aspects to be formulated in Social Choice Theory (See e.g. Moulin [1983], Peleg [1984] and Gibbard [1973]).

We start to investigate order morphisms. Once we have enough results on this subject, we investigate (a) by what conditions on the range neutrality is implied by the independence of irrelevant alternatives and Pareto-optimality and (b) by what conditions on the range the order morphisms can be extended to, e.g., $W_n(U)$. The following standard assumptions are made.

<u>Assumption 4.3.1</u>

Suppose N is a set of individuals, with $|N| = n$, $V \subseteq T(U)$ is a classified set of orderings and F is an order morphism from V_n to $V_2(U)$.

■

By theorem 4.1.10 $\hat{F}$ is a Pareto-optimal, neutral and independent of irrelevant alternatives complete welfare function from V_n to $V_2(U)$, the set of strongly complete relations.

If one is perfectly precize one should redefine all the decisiveness condition of §1.6 for complete welfare functions. Furthermore, in that case one should to reprove all the theorems proven in that section. As the reader might have already understood the definitions and theorems in §1.6 will be referred to as if they were also defined or proven for complete welfare functions. Obviously they only need a slight modification, e.g. F becomes $\hat{F}$ and $V_n(A)$ becomes V_n, in order to be appropriate for complete welfare functions. To save space we leave these modification to the reader's imagination.

Let us start by proving the final steps which yield dictatorship. By this approach the goal is clear.

<u>Theorem 4.3.2</u>

Assume (4.3.1).

Then $\hat{F}$ is weakly dictatorial, if for all S, T ε 2^N, such that qD($\hat{F}$,S) = U2 and qD($\hat{F}$,T) = U2, it holds that qD($\hat{F}$,S $\cap$ T) =U2, where U2 := $\{\langle x,y\rangle \in U \times U : |\{x,y\}| = 2\}$.

<u>Proof of theorem 4.3.2</u>

Suppose $\hat{F}$ is not weakly dictatorial and satisfies the above mentioned property.

Then for all i ε N B($\hat{F}$,{i}) $\neq$ U2.

Hence, by the independence of irrelevant alternatives and the neutrality of $\hat{F}$ it follows that:

For all i ε N, there is a S_i ε 2^N, such that i $\notin$ S_i and qD($\hat{F}$,S_i) = U2.

Hence, $\varphi = \cap \{S_i : i \in N\}$.

Since qD($\hat{F}$,S_i) =U2 for all i ε N it follows qD($\hat{F}$,φ) = U2.

Hence, $\hat{F}$ is not Pareto-optimal, which contradicts our assumptions. Therefore $\hat{F}$ is weakly dictatorial.

■

Hence, if $\hat{F}$ is not weakly dictatorial, then there are $S,T \in 2^N$ such that $qD(\hat{F},S) = U2 = qD(\hat{F},T)$ and $qD(\hat{F},S \cap T) \neq U2$. In that case it follows from the neutrality and independence of irrelevant alternatives of $\hat{F}$ that $qB(\hat{F},N - (S \cap T)) = U2$. We have just proven:

<u>Theorem 4.3.3</u>

Assume (4.3.1).

If $\hat{F}$ is not weakly dictatorial, then there are $S,T \in 2^N$, such that $qD(\hat{F},S) = qD(\hat{F},T) = qB(\hat{F},N -(S \cap T)) = U2$.

■

Theorem 4.3.3 will be used extensively. How this is done, is explained after the following two theorems, which have the same structure as the foregoing two. Therefore the proofs are left for the reader.

<u>Theorem 4.3.4</u>

Assume (4.3.1).

Then $\hat{F}$ is strongly dictatorial, if for all $S,T \in 2^N$, such that $qD(\hat{F},S) = U2$ and $qB(\hat{F},T) = U2$, it holds that $qD(\hat{F},S \cap T) = U2$.

■

<u>Theorem 4.3.5</u>

Assume (4.3.1).

If $\hat{F}$ is not strongly dictatorial, then there are $S,T \in 2^N$, such that: $qD(\hat{F},S) = qB(\hat{F},T) = qB(\hat{F},N -(S \cap T)) = U2$.

■

Especially theorem 4.3.5 and 4.3.3 are used in the proofs of impossibility theorems. The conclusions of having S, T and $N - (S \cap T)$ which have decisiveness properties as pointed out in theorem 4.3.3 and 4.3.5 lead to the construction of relations in $\hat{F}(L_n(U))$, which do not belong to the range of $\hat{F}$. Hence, those decisiveness assumptions about S, T and $N - (S \cap T)$ lead to contradictions concerning the range of $\hat{F}$. Therefore in a backwards reasoning $\hat{F}$ appears to be dictatorial.

In order to construct relations in $\hat{F}(L_n(U))$ on the basis of the decisiveness properties of S, T and $N - (S \cap T)$ it is necessary to know the interaction between decisive coalitions and $\hat{F}(L_n(U))$. Theorem 4.3.6 clarifies this interaction and makes it possible to deduce contradictions from decisiveness properties of S, T and N - (S ∩ T) as pointed out above.

Theorem 4.3.6

Let $\hat{F}$ from $L_n(U)$ to $V_2(U)$ be an independent of irrelevant alternatives complete welfare function on N, with |N| = n. Furthermore, let $R_A \in V_2(U)$. Then $\tilde{R}_A \in \hat{F}(L_n(U))$, iff there is a labeling $\underset{\sim}{l} : \bar{n}(A \times A)_A \to 2^N$ such that:

4.3.6.1 for all $\langle a,b \rangle \in \bar{a}R_A : \langle a,b \rangle \in qD(\hat{F}, l(\langle a,b \rangle))$,

4.3.6.2 for all $\langle a,b \rangle \in \overline{ns\tilde{R}_A} : \langle a,b \rangle \in qB(\hat{F}, l(\langle a,b \rangle))$,

4.3.6.3 for all $\langle a,b \rangle \in \bar{n}(A \times A)_A : l(\langle a,b \rangle) \cup l(\langle b,a \rangle) = N$ and $l(\langle a,b \rangle) \cap l(\langle b,a \rangle) = \varphi$, and

4.3.6.4 for all $\langle a,b \rangle, \langle b,c \rangle, \langle a,c \rangle \in \bar{n}(A \times A)_A$: $l(\langle a,b \rangle) \cap l(\langle b,c \rangle) \subseteq l(\langle a,c \rangle)$.

Proof of theorem 4.3.6

(only if) Let $\tilde{R}_A \in \hat{F}(L_n(U))$. Then such a labeling should be constructed.

Obviously there is a $r_A \in L_n(U)$, with $\hat{F}(r_A) = \tilde{R}_A$.

For all $\langle a,b \rangle \in \bar{n}(A \times A)_A$, define

$l(\langle a,b \rangle) := \{i \in N : a > b : R^i_A\}$.

Then (4.3.6.3) follows from the fact that $r_A \in L_n(U)$ and the definition of $\hat{F}$. (4.3.6.4) follows from the fact that the components of r_A are transitive. (4.3.6.1) and (4.3.6.2) follow from the fact that $\hat{F}$ is independent of irrelevant alternatives and the definitions of qD, qB and l.

Hence, evidently we have such a labeling.

(if) Let l be such a labeling.

Take $R^i_A \in V_2(U)$, for all $i \in N$, defined as follows:

$R^i_A := \bar{r}\langle\{\langle a,b \rangle \in A \times A : i \in l(\langle a,b \rangle)\}, A\rangle$.

By (4.3.6.4) $R^i_A \in T(U)$. By (4.3.6.4) it is transitive. Hence, $R^i_A \in L(U)$ for all $i \in N$ and $r_A \in L_n(U)$. Moreover, by the definition of qD and qB it is evident that $\hat{F}(r_A) = \tilde{R}_A$. ■

Theorem 4.3.6 states that $\tilde{R}_A \in \hat{F}(L_n(U))$, iff we can label the pairs of $\tilde{R}_A$ with decisive coalitions according to (4.3.6.1), (4.3.6.2), (4.3.6.3) and (4.3.6.4). Furthermore, the proof indicates how a profile is constructed when the labeling is known and vica versa.

By virtue of theorem 4.3.6 and the decisiveness properties of S, T and N - (S ∩ T) we are able to construct elements in $\hat{F}(L_n(U))$ which, for instance, are not $\langle \bar{i}^2, \bar{i} \rangle$-classifiably transitive.

Theorem 4.3.6 supplies us with a pictorial support in the construction of images of $\hat{F}$, a support that is demonstrated below.

First a convention is introduced.

By $\underset{x}{.} \xleftarrow[M]{} \underset{y}{.}$ the following situation is represented:

$l(\langle x,y \rangle) = N - M$ and $l(\langle y,x \rangle) = M$.

Hence, $$\begin{tikzcd} x \arrow[r, "S"] & y \arrow[r, "T"] & z \arrow[ll, bend left, "M"] \end{tikzcd}$$ corresponds with the following labeling: $l(\langle x,y \rangle) = S$, $l(\langle y,x \rangle) = N - S$, $l(\langle y,z \rangle) = T$,

$l(\langle z,y \rangle) = N - T$, $l(\langle z,x \rangle) = M$, $l(\langle x,z \rangle) = N - M$.

Suppose for instance that $S \cup T \cup M = N$ and $S \cap T \cap M = \varphi$ then the following Condorcet-profile corresponds with this picture:

$x > y > z \; : \; R^i_{\{x,y,z\}} \quad i \in S \cap T$,

$y > z > x \; : \; R^i_{\{x,y,z\}} \quad i \in T \cap M$, and

$z > x > y \; : \; R^i_{\{x,y,z\}} \quad i \in M \cap S$.

After this short pictorial explanation we will prove the following lemma, which states that various decisiveness properties lead to various intransitivities of $\hat{F}(L_n(U))$.

Lemma 4.3.7

Assume (4.3.1). Furthermore, suppose $M,S,T \in 2^N$, such that $N = M \cup S \cup T$ and $M \cap S \cap T = \varphi$, and k is an integer, with $k \geq 2$.

4.3.7.1 If $qD(\hat{F},S) = qD(\hat{F},T) = qD(\hat{F},M) = U2$, then there is a $R_A \in \hat{F}(L_n(U))$, which is not $\langle \bar{a}^k, \bar{i}^{k-1}\rangle$-classifiably transitive.

4.3.7.2 If $qD(\hat{F},S) = qD(\hat{F},T) = qB(\hat{F},M) = U2$, then there is an $R_A \in \hat{F}(L_n(U))$, which is not $\langle \bar{a}^k, \overline{ra}^{k-1}\rangle$-classifiably transitive.

4.3.7.3 If $qB(\hat{F},S) = qB(\hat{F},T) = qD(\hat{F},M) = U2$, then there is a relation $R_A \in \hat{F}(L_n(U))$, which is not $\langle \bar{i}^k, \bar{i}^{k-1}\rangle$-classifiably transitive.

4.3.7.4 If $qD(\hat{F},S) = qD(\hat{F},T) = qB(\hat{F},M) = U2$, then there is an $R_A \in \hat{F}(L_n(U))$, which is not $\langle \bar{a}^l\bar{i}\bar{a}^m, \overline{ra}^2\rangle$-classifiably transitive, where $l \geq 1$ and $m \geq 1$.

4.3.7.5 If $qD(\hat{F},S) = qD(\hat{F},T) = qB(\hat{F},M) = U2$, then there is an $R_A \in \hat{F}(L_n(U))$, which is not $\langle \bar{a}^l\bar{i}\bar{a}^m, \bar{i}\rangle$-classifiably transitive, where $l + m \geq 1$.

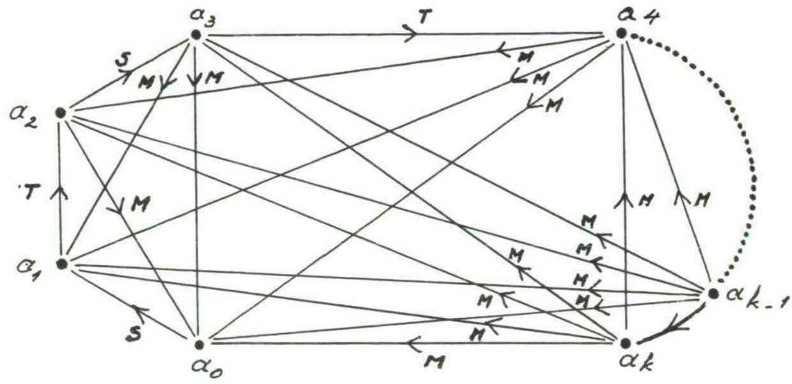
Proof of lemma 4.3.7

(4.3.7.1),(4.3.7.2) and (4.3.7.3) Let $A = \{a_0,a_1,a_2,\ldots,a_k\}$.

Take $l : \bar{n}(A \times A)_A \to 2^N$ as follows:

$$l(\langle a_i,a_j\rangle) := \begin{cases} S, & \text{iff } j = i + 1 \text{ and } i \text{ is even,} \\ T, & \text{iff } j = i + 1 \text{ and } i \text{ is odd,} \\ M, & \text{iff } i > j + 1, \\ N - S, & \text{iff } j + 1 = i \text{ and } j \text{ is even,} \\ N - T, & \text{iff } j + 1 = i \text{ and } j \text{ is odd, and} \\ N - M, & \text{iff } j + 1 > i. \end{cases}$$

Diagram:



Then l is a labeling for which obviously (4.3.6.2) and (4.3.6.4) hold. Applying theorem 4.3.6 to:
the premise of (4.3.7.1) it follows that $R_A \in \hat{F}(L_n(U))$,
the premise of (4.3.7.2) it follows that $R'_A \in \hat{F}(L_n(U))$, and
the premise of (4.3.7.3) it follows that $R''_A \in \hat{F}(L_n(U))$,
where $R_A := \bar{r}\langle\{\langle a_i,a_j\rangle \in A \times A : l(\langle a_i,a_j\rangle) \in \{S,T,M\}\},A\rangle$,
$\bar{r}\langle\{\langle a_i,a_j\rangle \in A \times A : l(\langle a_i,a_j\rangle) = M\},A\rangle \subseteq R'_A$,
$\langle\{\langle a_i,a_j\rangle \in A \times A : l(\langle a_i,a_j\rangle) \in \{S,T\}\},A\rangle \subseteq \bar{a}R'_A$,
$R_A \subseteq R''_A$ and $\langle\{\langle a_i,a_j\rangle \in A \times A : l(\langle a_i,a_j\rangle) = M\},A\rangle \subseteq \bar{a}R''_A$.
So the relation indicated by the diagram is equal to R_A and is contained in R'_A as well as in R''_A. Now note that $\langle a_0,a_1,a_2,\ldots,a_k\rangle$ is a path along R_A, R'_A and R''_A of type $\bar{a}^k$, $\bar{a}^k$ and $\bar{i}^k$ respectively, which cannot be cut short along R_A, $\bar{a}R'_A$ and R''_A
respectively. Hence, R_A is not $\langle\bar{a}^k,\bar{i}^k\rangle$-transitive, R'_A is not $\langle\bar{a}^k,\overline{ra}^{k-1}\rangle$-transitive and R''_A is not $\langle\bar{i}^k,\bar{i}^{k-1}\rangle$-transitive.

(4.3.7.4) Since $\langle\bar{a}^1\bar{i}\bar{a}^m,\overline{ra}^2\rangle$-classifiable transitivity implies $\langle\bar{a}^{1+m+1},\overline{ra}^2\rangle$-transitivity we are done by (4.3.7.2).

(4.3.7.5) The labeling l establishes again such an intransitivity.
This completes the proof.

■

Before deducing some immediate consequences of the foregoing theorem it is pointed out that $\langle \bar{a}^k, \bar{i}^{k-1} \rangle$-transitivity is a very weak transitivity condition. It means that a path π of type $\bar{a}^k$ can be cut short by a path of type $\bar{i}^t$ for any $t \leq k-1$. Hence, such a short cut is established, whenever there are two alternative a,b on π which are not succeeding each other on π and $\langle a,b \rangle$ is in
the relation. Note that this transitivity does not exclude circuits. $\langle \bar{a}^k, \overline{ra}^{k-1} \rangle$-transitivity is stronger than $\langle \bar{a}^k, \bar{i}^{k-1} \rangle$-transitivity, but still weak when compared to frequently used transitivity conditions for the range of a welfare function. Of course $\langle \bar{i}^k, \bar{i}^{k-1} \rangle$-transitivity is again stronger.

It is perhaps useful to mention that R_A is a tournament on k+1 alternatives with a maximum number of 3-cycles. (The proof of this fact is standard in Graph Theory).

This results in the following theorem.

Theorem 4.3.8

Assume (4.3.1).

4.3.8.1 If there is an integer $k \geq 2$, such that all $R_A \in \hat{F}(V_n)$ are $\langle \bar{a}^k, \overline{ra}^{k-1} \rangle$-classifiably transitive, then $\hat{F}$ is weakly dictatorial.

4.3.8.2 If there is an integer $k \geq 2$, such that all $R_A \in \hat{F}(V_n)$ are $\langle \bar{i}^k, \bar{i}^{k-1} \rangle$-classifiably transitive, then $\hat{F}$ is strongly dictatorial.

Proof of theorem 4.3.8

(4.3.8.1) Suppose $\hat{F}$ is not weakly dictatorial.

By theorem 4.3.3 there are $S,T \in 2^N$ such that $qD(\hat{F},S) = qD(\hat{F},T) = qB(\hat{F},M) = U2$, with $M := N - (S \cap T)$.

Hence, by (4.3.7.2) it is easy to deduce a contradiction.

Therefore $\hat{F}$ is weakly dictatorial.

(4.3.8.2) Suppose $\hat{F}$ is not strongly dictatorial.

Then by theorem 4.3.5 and (4.3.7.3) we can similarly deduce a contradiction.

Hence, $\hat{F}$ is strongly dictatorial.

■

It is clear that the proposed transitivities are not the only ones which lead to dictatorship. Of course, besides the new results of theorem 4.3.8 one is able to prove other new impossibility theorems. To give the reader an idea it is similarly provable that if the range $\hat{F}(V_n) \subseteq T(U)$ and is $<\bar{a}^k,\bar{i}^k>$-transitive, then $\hat{F}$ is strongly dictatorial. If the range $\hat{F}(V_n)$ is $<\underbrace{\bar{a}\bar{s}\bar{a}\bar{s}......}_{\text{k-symbols}},\overline{ra}^{k-1}>$-transitive, then $\hat{F}$ is strongly dictatorial. If the range $\hat{F}(V_n)$ is $<\underbrace{\bar{a}\bar{s}\bar{a}......}_{\text{k-symbols}},\bar{i}^{k-1}>$-transitive, then $\hat{F}$ is weakly dictatorial. And so on and so on.

All these impossibilities can be deduced by using theorem 4.3.3 or 4.3.5 and constructing relations in $\hat{F}(L_n(U))$, which have not the required transitivity properties. Of course these relations are constructed by virtue of theorem 4.3.6. Since these constructions are simular to those of the previous theorems this work is not done here.

The neutrality condition imposed on the welfare functions in theorem 4.3.8 is not frequently used in other impossibility theorems. Therefore, it is proven that various transitivity conditions imposed on the range of a welfare function along with the Pareto-optimality and the independence of irrelevant alternatives, imply the neutrality property. Hence, the neutrality property is often implicitly imposed on welfare functions. The fact is not a new result in Social Choice Theory. (See Barthélémy [1983]). We will show that several standard results in Social Choice Theory can be deduced in this context.

<u>Theorem 4.3.9</u>

Let $\hat{F}$ be a complete welfare function from $L_n(U)$ to $V_2(U)$ which is Pareto-optimal and independent of irrelevant alternatives.

4.3.9.1 If for all $R_A \in \hat{F}(L_n(U))$, R_A is $<\bar{a}^l\bar{i}\bar{a}^m,\overline{ra}^2>$-classifiably transitive, then $\hat{F}$ is neutral, where $l \geq 1$ and $m \geq 1$.

4.3.9.2 If for all $R_A \in \hat{F}(L_n(U))$, R_A is $<\bar{a}^l\bar{i}\bar{a}^m,\bar{i}>$-classifiably transitive, then $\hat{F}$ is neutral, where $l + m \geq 1$.

4.3.9.3 If for all $R_A \in \hat{F}(L_n(U))$, R_A is $<\bar{a}^k,\bar{a}>$-classifiably transitive, then $\hat{F}$ is neutral, where $k \geq 2$.

Proof of theorem 4.3.9

(4.3.9.1) Suppose $\hat{F}$ is as in (4.3.9.1).

There are three steps.

Step 1 If $\langle a,b\rangle \in qD(\hat{F},S)$, then $\langle a,x\rangle \in qD(\hat{F},S)$, for all $S \in 2^N$ and all a,b,x, such that $|\{a,b,x\}| = 3$.

Suppose $\langle a,b\rangle \in qD(\hat{F},S)$, $x \in A - \{a,b\}$ and $\langle a,x\rangle \notin qD(\hat{F},S)$. Then by the independence of irrelevant alternatives it follows that $\langle x,a\rangle \in qB(\hat{F},N-S)$.

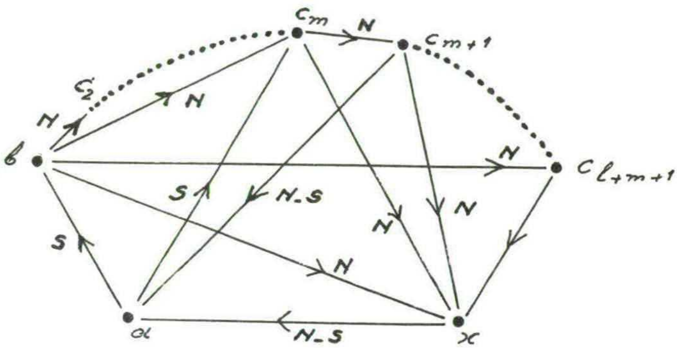
Take the following profile $r_A \in L_n(A)$:

$a\ b\ c_2\ c_3\ \ldots\ldots\ c_m\ c_{m+1}\ \ldots\ldots\ c_{m+l+1}\ x\ :\ R_A^i$ for $i \in S$, and

$b\ c_2\ c_3\ \ldots\ldots\ c_m\ c_{m+1}\ \ldots\ldots\ c_{m+l+1}\ x\ a\ :\ R_A^i$ for $i \in N-S$,

where $A = \{a,b,x,c_2,c_3,\ldots,c_{m+l+1}\}$.

Observe the following picture of the labeling according to r_A.



Then $\langle c_{m+1},\ldots\ldots,c_{m+l+1},x,a,b,c_2,c_3,\ldots,c_m\rangle$ is a path along $\hat{F}(r_A)$ of type $\overline{a}^{-1}\overline{ia}^{m}$. Since it can be cut short by a path of type $\overline{ra}^2$, we have $c_{m+1} > a : \hat{F}(r_A)$ and $a > c_m : \hat{F}(r_A)$.

Now $\langle c_{m+1},a\rangle \in qD(\hat{F},N-S)$ and $\langle a,c_m\rangle \in qD(\hat{F},S)$ by the independence of irrelevant alternatives of $\hat{F}$. But similarly it is deducible that $\langle a,c_{m+1}\rangle \in qD(\hat{F},S)$. Hence, we have a contradiction.

Step 2 If $\langle a,b\rangle \in qD(\hat{F},S)$, then $\langle x,a\rangle \in qD(\hat{F},S)$, for all $S \in 2^N$ and all a,b,x, such that $|\{a,b,x\}| = 3$.

The proof is similar to step 1.

Step 3 $qD(\hat{F},S) = U2$, iff there are $a,b \in U$, $a \neq b$, such that $\langle a,b\rangle \in qD(\hat{F},S)$, for all $S \in 2^N$.

The proof is standard in literature (See e.g. Barthélémy [1983]). Step 1 and step 2 and $|U| \geq 3$ establish that $\bar{r}(qD(\hat{F},S))$ is an equivalence relation, which is complete if $qD(\hat{F},S)$ has any non diagonal pair.

Now by step 3 and theorem 1.6.2.5 we are done.

(4.3.9.2) and (4.3.9.3) are similar to (4.3.9.1).

■

Note that many transitivity conditions imposed on the range of a welfare function in literature are stronger than those of theorem 4.3.9. Hence, in literature often implicitly neutrality is imposed on welfare functions. Furthermore, the steps as pointed out in (4.3.9.1) occur as intermediate results in many impossibility results.

We will now deduce some results which are well-known.

Theorem 4.3.10

Suppose $\hat{F}$ is a Pareto-optimal and independent of irrelevant alternatives welfare function from V_n to $V_2(U)$, with $n \geq 1$, and $V \subseteq V_2(U)$ is a classified set of orderings.

4.3.10.1 If $\hat{F}(V_n)$ is contained in the set of $\langle \bar{a}^{l}\bar{s}\bar{a}^{m},\overline{ra}^{t}\rangle$-transitive and $\langle \bar{a}^{l}\bar{a}\bar{a}^{m},\overline{ra}^{t}\rangle$-transitive relations, where $l + m \geq t \geq 1$ and $t \leq 2$, then $\hat{F}$ is strongly dictatorial.

4.3.10.2 If $\hat{F}(V_n)$ is contained in the set of $\langle \bar{a}^{k},\bar{a}\rangle$-transitive relations, where $k \geq 2$, then $\hat{F}$ is oligarchic, i.e., there is a $S \subseteq N$, such that $D(\hat{F},S) = U2$ and $B(\hat{F},\{i\}) = U2$ for all $i \in S$.

Proof of theorem 4.3.10

(4.3.10.1) Suppose (4.3.10.1).

Since $\langle \bar{a}^{l}\bar{s}\bar{a}^{m},\overline{ra}^{t}\rangle$-transitivity and $\langle \bar{a}^{l}\bar{a}\bar{a}^{m},\overline{ra}^{t}\rangle$-transitivity is equivalent to , $\langle \bar{a}^{l}\bar{i}\bar{a}^{m},\overline{ra}^{t}\rangle$-transitivity, it follows by (4.3.9.1) and (4.3.9.2) that $\tilde{F}$ is neutral, with $\tilde{F} := \hat{F}|_{L_n(U)}$.

By theorem 4.3.7.4, 4.3.7.5 and 4.3.3 $\tilde{F}$ is weakly dictatorial.

Hence, there is an individual $\hat{\imath} \in N$, such that $qB(\hat{F},\{\hat{\imath}\}) = U2$.

The following two steps complete the proof of (4.3.10.1).

<u>Step 1</u> For all $S \in 2^N$: if $qB(\hat{F},S) = U2$, then $qD(\hat{F},S) = U2$.

<u>Proof of step 1</u> By the neutrality of $\tilde{F}$ it is sufficient to deduce a contradiction from the following assumption:

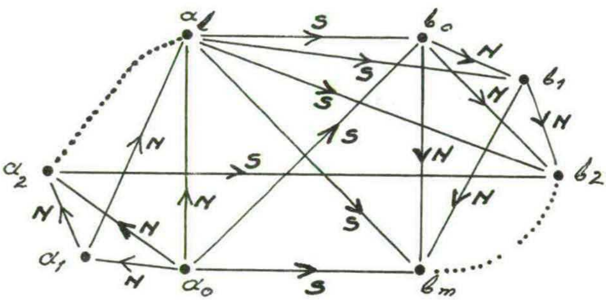
Suppose $qB(\hat{F},S) = qB(\hat{F},N-S)$.

Take the following profile $r_A \in L_n(A)$:

$a_0\ a_1\ a_2 \ldots a_l\ b_0\ b_1 \ldots b_m : R_A^i$ for $i \in S$, and

$b_0\ b_1 \ldots b_m\ a_0\ a_1\ a_2 \ldots a_l : R_A^i$ for $i \in N-S$, where $\{a_0,\ldots,a_l,b_0,\ldots,b_m\} = A$.

We have the following diagram:



Hence, $\langle a_0,\ldots,a_l,b_0,\ldots,b_m\rangle$ is a path of type $\bar{a}^{l}\bar{s}\bar{a}^{m}$ that cannot be cut short by a path of type $\overline{ra}^{t}$.

This contradicts our transitivity assumptions.

<u>Step 2</u> For all $S \in 2^N$: if $qD(\hat{F},S) = U2$, then $D(\hat{F},S) = U2$.

<u>Proof of step 2</u> Let $S \in 2^N$ be such that $qD(\hat{F},S) = U2$.

Let $a,b \in U$ such that $a \neq b$.

Suppose $\hat{r}_A \in V_n$ with $a > b : \hat{R}_A^i$, for $i \in T_1$, $a \sim b : \hat{R}_A^i$ for $i \in T_2$ and $b > a : \hat{R}_A^i$ for $i \in T_3$, where $T_1 \cup T_2 \cup T_3 = N$ and $S \subseteq T_1$.

It suffices to prove that $a > b : \hat{F}(\hat{r}_A)$.

Suppose not $a > b : \hat{F}(\hat{r}_A)$. Then $b \geq a : \hat{F}(\hat{r}_A)$.

Take the following profile $\tilde{r}_A$, which by our knowledge of minimal extensions of L(U) is in V_n:

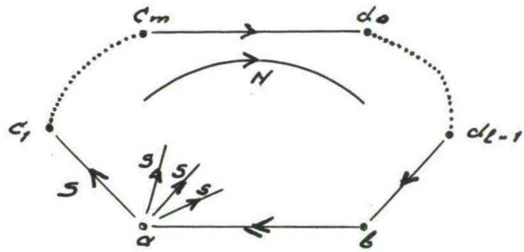
$a\ c_1\ c_2 \ldots c_m\ d_0\ d_1 \ldots d_{l-1}\ b \quad : \tilde{R}_A^i \quad i \in S,$

$c_1\ c_2 \ldots c_m\ d_0\ d_1 \ldots d_{l-1}\ a\ b \quad : \tilde{R}_A^i \quad i \in T_1 - S,$

$c_1\ c_2 \ldots c_m\ d_0\ d_1 \ldots d_{l-1}\ (a\ b) : \tilde{R}_A^i \quad i \in T_2$, and

$c_1\ c_2 \ldots c_m\ d_0\ d_1 \ldots d_{l-1}\ b\ a \quad : \tilde{R}_A^i \quad i \in T_3.$

Then we have the following diagram:



From the Pareto-optimality, the independence of irrelevant alternatives and the fact that $qD(\hat{F},S) = U2$, it follows that $\langle d_0,d_1\ldots d_{m-1},b,a,c_1,c_2\ldots c_l\rangle$ is a path along $\hat{F}(\tilde{r}_A)$ of type $\bar{a}^{l}\,\overline{ia}^{m}$ that cannot be cutted short by a path of type $\overline{ra}^{t}$.

This completes the prove of step 1 and 2.

(4.3.10.2) Suppose (4.3.10.2).

By theorem 4.3.9.3, 4.3.7.2 and 4.3.3 it follows that $\tilde{F}$ is weakly dictatorial, where $\tilde{F} := \hat{F}|_{L_n(U)}$. So there are $i \in N$ with $qB(\hat{F},\{i\}) = U2$.

Let $S := \{i \in N : qB(\hat{F},\{i\}) = U2\}$.

Again the proof is completed by the following step.

<u>Step 3</u> For all $S \in 2^N$: if $qB(\tilde{F},S) = U2$, then $B(\hat{F},S) = U2$.

<u>Proof of step 3</u> Suppose $B(\hat{F},S) \neq U2$.

Then there is a profile $\hat{r}_A$ and a pair $a,b \in U$, $a \neq b$, such that:

$a > b : \hat{R}^i_A$ for $i \in T_1$,

$a \sim b : \hat{R}^i_A$ for $i \in T_2$,

$a < b : \hat{R}^i_A$ for $i \in T_3$, $S \subseteq T_1$ and $a < b : \hat{F}(\hat{r}_A)$.

Take $\tilde{r}_A$ in $V_n(A)$ as follows:

$b\ a\ c_2\ c_3\ldots c_k \quad : \tilde{R}^i_A$ for $i \in T_3$,

$(a\ b)\ c_2\ c_3\ldots c_k : \tilde{R}^i_A$ for $i \in T_2$,

$a\ b\ c_2\ c_3\ldots c_k \quad : \tilde{R}^i_A$ for $i \in T_1 - S$, and

$a\ c_2\ c_3\ldots c_k\ b \quad : \tilde{R}^i_A$ for $i \in S$.

Again using the independence of irrelevant alternatives, the Pareto-optimality of $\hat{F}$ and the fact that $qB(\tilde{F},S) = qB(\hat{F},S) = U2$ it follows from $b > a : \hat{F}(\hat{r}_A)$ that

$\langle b,a,c_2,c_3,\ldots c_k\rangle$ is a path of type $\bar{a}^k$ along $\hat{F}(\tilde{r}_A)$ and $c_k \geq b : \hat{F}(\tilde{r}_A)$. Hence, this path cannot be cutted short by a path of type $\bar{a}$ and therefore $\hat{F}(\tilde{r}_A)$ is not $\langle\bar{a}^k,\bar{a}\rangle$-classifiably transitive.

This contradicts our assumptions and proves step 3.

By step 3 it follows that $\hat{S} = \{i \in N : B(\hat{F},\{i\}) = U2\}$ and therefore $B(\hat{F},\hat{S}) = U2$.

From the definition of $\hat{S}$ it follows that for all $i \in N - \hat{S}$ $qB(\hat{F},\{i\}) \neq U2$. Hence, for all $i \in N - \hat{S}$ there are $a,b \in \underset{\sim}{U}$, $a \neq b$, such that $\langle a,b\rangle \in qD(\tilde{F},N-\{i\})$. By the neutrality of $\tilde{F}$ it follows that for all $i \in N - \hat{S}$, $qD(\tilde{F},N-\{i\}) = U2$.

Take $X \subseteq N$, such that $\hat{S} \subseteq X \subset N$, $qD(\tilde{F},X) = U2$ and X is minimal with those properties,

Suppose $\hat{S} \subset X$.

Then there is an $i \in X - \hat{S} \subseteq N - \hat{S}$.

Hence, $qD(\tilde{F},N-\{i\}) = qD(\tilde{F},X) = U2$.

Now by (4.3.7.2) and the neutrality of $\tilde{F}$ we have $qB(\tilde{F},N - ((N-\{i\}) \cap X)) = \varphi$.

Therefore $qD(\tilde{F}, X - \{i\}) = qD(\tilde{F},(N-\{i\}) \cap X) = U2$.

This contradicts our assumptions about X.

Hence, $X = \hat{S}$ and $qD(\tilde{F},\hat{S}) = U2$.

Similar to step 2 one can prove that $D(\hat{F},\hat{S}) = U2$ is implied by $D(\tilde{F},\hat{S}) = U2$.

So $\hat{F}$ is oligarchic.

■

Let us compare the results of this section with several well-known impossibility results. As stated before, theorem 4.3.8 is new because of the new transitivity concept, introduced in chapter two. However, for special choices of k, (4.3.8) one has well-known similar results in literature, which even look stronger, because neutrality is not used in those impossibility theorems. We will discusse these facts somewhat further.

For instance, if $k = 2$ in (4.3.8.2) by (4.1.10), it follows for every welfare function F from $V_n(A)$ to $W(A)$, with $|A| \geq 3$: if F is neutral, independent of irrelevant alternatives and Pareto-optimal, then F is strongly dictatorial. The fact $|A| \geq 3$ should be introduced since only for $|A| \geq 3$ $\langle\bar{i}^2,\bar{i}\rangle$-transitivity is a meaningful property. Furthermore, V is any classified set of orderings and W(U) is of course the set in $V_2(U)$ of $\langle \bar{i}^2, \bar{i} \rangle$-transitive relations. Now similar results are known in literature with two differences. In literature V(A) is often chosen from {L(A), W(A), Q(A)} and neutrality is not imposed on F. If done so we get what is often referred to as Arrow's impossibility theorem (See: Moulin [1983], Kelly [1978]).

Note that for relations in $V_2(U)$, $\langle \bar{i}^2, \bar{i} \rangle$-transitivity is equivalent to $\langle \bar{a}\bar{i}\bar{a}, \bar{a} \rangle$-transitivity together with $\langle \bar{a}\bar{a}, \bar{a} \rangle$-transitivity. So by (4.3.10.1) it follows that for all F from $V_n(A)$ to W(A), with $|A| \geq 3$: if F is independent of irrelevant alternatives and Pareto-optimal, then F is strongly dictatorial.

Hence, (4.3.10.1) implies the well-known Arrow-theorem.

Similarly (4.3.10.1) and (4.1.10) imply that for all independent of irrelevant alternatives and Pareto-optimal F from $V_n(A)$ to $V_2(A)$, with $|A| \geq l + m + 2$,
$F(V_n(A)) \subseteq \{R_A \in V_2(A) : R_A$ is $\langle \bar{a}^l \bar{s} \bar{a}^m, \overline{ra}^t \rangle$-transitive and $\langle \bar{a}^{l+m+1}, \overline{ra}^t \rangle$-transitive$\}$, $l + m \geq t \geq 1$ and $t \leq 2$, F is strongly dictatorial.

This result is a generalization of impossibility theorems in Wilson [1972] ($l = m = t = 1$), in Blau [1979] ($l = 2$ and $t = 1$) and in Blair & Pollack [1979] ($t = 1$).

Similarly we can deduce another theorem as in Blair & Pollack [1979] from (4.3.10.2).

From these observations it will be clear that in our framework a lot of results in literature can be generalized. Many theorems are left undiscussed, especially those concerning acyclic collective preferences (See e.g., Ferejohn & Grether [1974] and Blair & Pollack [1982]). But we will end this type of research because we are convinced that these results can be treated in a similar way. They would only lead to more cases with specific transitivity or acyclicity conditions, together with some extra assumptions, that lead to dictatorship. No new insight would be gained, nor would new methods be revealed to prove these impossibility theorems.

Now other relations between domain and range of an order morphism are deduced. Since an order morphism is equivvalent to a welfare function with "nice" properties (corollary 4.1.11), these relations give us more insight in Social Choice Theory. It will be proven that either the range is very complex compared to the domain or the order morphism is simply (dictatorial). In order to explain this, consider (4.3.8.1). It is not difficult to prove, similarly to (4.3.10.2), that if

$\hat{F} : L_n(U) \to \{R_A \in V_2(U) : R_A$ is $\langle \bar{a}^k, \overline{ra}^{k-1}\rangle$-transitive$\}$ is an order morphism, then $\hat{F}$ is oligarchic.
Here $S := \{i \in N : B(\hat{F},\{i\}) = U2\}$ is the oligarchy.

So $\hat{F} : L_n(U) \to \{R_A \in V_2(U) : R_A$ is $\langle \bar{a}^k, \overline{ra}^{k-1}\rangle$-transitive$\}$ has the following form for all $r_A \in L_n(U)$: $\hat{F}(r_A) = \cap \{R^i_A : i \in S\}$. $\hat{F}$ is not very complicated. (Consequently it cannot describe decision rules used in "real life"). To avoid this simplicity of $\hat{F}$ one has to choose another relationship, between domain and range of $\hat{F}$, as e.g., $\langle \bar{a}^2, \bar{a}\rangle$-transitivity or $\langle \bar{a}^k, \bar{a}^{k-1}\rangle$-transitivity.

Note that all the impossibility theorems deduced above, as well as those in literature, have the same interpretation. Furthermore, in all those theorems transitivity or acyclicity conditions are imposed on the range. An interesting question is now: Is there any (meaningful) transitivity condition on the image $\hat{F}(L_n(U))$ such that $\hat{F}$ has not such a simple structure?

Although all the foregoing theorems suggest that such a transitivity condition does not exist, the question is at least yet for us too difficult to answer. On the other hand, the conjecture is negative, because by theorem 4.3.8 $\langle \bar{a}^k, \overline{ra}^{k-1}\rangle$-transitivity is too strong. If $\langle \bar{a}^k, \bar{i}^{k-1}\rangle$-transitivity were also too strong one could argue that the answer is negative, since the transitivity is very weak. For a small number of individuals, i.e., $|N| = 3$, there is such an impossibility.

Theorem 4.3.11

Assume (4.3.1), such that n = 3.
If $\hat{F}(L_n(U)) \subseteq \{R_A \in V_2(U) : R_A$ is $\langle \bar{a}^k, \bar{i}^{k-1}\rangle$-transitive$\}$ for some $k \geq 2$, then $\hat{F}$ is weakly dictatorial.

Proof of theorem 4.3.11

Suppose $\hat{F}$ is not weakly dictatorial.

Then for all $i \in N$ $qD(\hat{F}, N - \{i\}) \neq \varphi$.

By the neutrality and the independence of irrelevant alternatives of $\hat{F}$ it follows now that for all $i \in N$: $qD(\hat{F}, N - \{i\}) = U2$.

Since $|N| = 3$, there are precisely three such coalitions, whose intersection is empty and whose union is N. Hence, by (4.3.7.1) we have a contradiction.

Therefore, $\hat{F}$ is weakly dictatorial.

■

Again emphasizing the fact that $\langle \bar{a}^k, \bar{i}^{k-1} \rangle$-transitivity is a very weak condition, since if this does not hold one cannot lay any meaningful transitivity condition on the asymmetric part of a (complete) relation, and waiting for generalizations of (4.3.11) we come to the following more general question:

Note that $\hat{F}(L_n(U))$ is a classified set of orderings, how 'big', compared to L(U) should $\hat{F}(L_n(U))$ be, such that $\hat{F}$ is not a simple dictatorship?

We will deduce some new impossibility results, i.e. this 'bigness' is very great in a special context.

First the following theorem is proven.

Theorem 4.3.12

4.3.12.1 Assume (4.3.1), such that there are $W_0, W_1, W_2, ..W_k \subset T(U)$ and $W_0 = V$, $W_k = \hat{F}(V_n)$ and $W_i = W_{i+1}$ or $W_i \underset{m}{\subsetneq} W_{i+1}$, for all $i \in \{1,2,\ldots,k\}$.

4.3.12.2 Then $\hat{F}$ is strongly dictatorial.

■

Before proving theorem 4.3.12 we have the following remarks:

+ By the Pareto-optimality it follows that $V \subseteq \hat{F}(V_n)$. So $V \subseteq \hat{F}(V_n)$ is not only implied by the assumptions about W_0 up to W_k.
+ It is a symplification to assume that $\hat{F}(V_n) \subseteq T(U)$. We do not know what happens if this is dropped.
+ T(U) is the set of all complete and asymmetric relations. $\hat{F}(V_n) \subset T(U)$ means that the classified set of relations $\hat{F}(V_n)$ has some propably very weak extra properties.

+ $V \underset{m}{\subsetneq} W_1 \underset{m}{\subsetneq} \ldots \underset{m}{\subsetneq} W_{k-1} \underset{m}{\subsetneq} \hat{F}(V_n)$ means that only a finite number of extra relations not in V is needed to construct an abitrary relation in $\hat{F}(V_n)$, by the construction tools substitution, concatenation, restriction, conversion and permutation.

Hence, theorem 4.3.12 states that it is impossible to have non-simple order morphisms from V_n to W, where V and W are classified sets of tournaments, both strictly contained in $T(U)$ and W can be "constructed" from V in a finite way.

We will prove this theorem by some intermediate results.

<u>Lemma 4.3.13</u>

Assume (4.3.12.1).

If $R_X \in W_k$ and $Y \subseteq X$, such that $0 < |Y| \leq |X| - k$, and R_X is irreducible, then $R_{X|_Y} \in W_0$.

<u>Proof of lemma 4.3.13</u>

Since $W_t = \Sigma_4\Sigma_6(W_{t-1} \cup \{R_Z^t, \bar{v}R_Z^t\})$, such that $R^t_{Z|_A}$, $\bar{v}R^t_{Z|_A} \in W_{t-1}$, for all $0 < |A| < |Z|$ and $A \subseteq Z$, (this follows by theorem 2.4.5). This lemma follows by a simple induction on $t \in \{1,2,\ldots,k\}$.

■

Essentially theorem 4.3.12 is proven by deducing a contradiction from the following assumption:

<u>Assumption 4.3.14</u>

Assume (4.3.1).

Suppose furthermore there are $W_0, W_1, W_2, \ldots W_k \subset T(U)$, such that $W_0 = V$, $W_k = \hat{F}(V_n(U))$ and for all $i \in \{1,2,\ldots,k\}$ $W_i = W_{i+1}$ or $W_i \underset{m}{\subsetneq} W_{i+1}$, and $\hat{F}$ is not strongly dictatorial.

■

By theorem 4.3.5 and $\hat{F}(V_n) \subseteq T(U)$ we have:

<u>Lemma 4.3.15</u>

Assume (4.3.14).

Then there are $S, T, M \in 2^N$, such that $S \cap T \cap M = \varphi$, $S \cup T \cup M = N$, and $qD(\hat{F},S) = qD(\hat{F},T) = qD(\hat{F},M) = U2$.

■

We can prove now the first astonishing result.

<u>Lemma 4.3.16</u>

Assume (4.3.14).

Then for all $l \in \{0,1,2,\ldots,k\}$: if $T_{m,l}(U) \subseteq \hat{F}(V_n)$ for all $m \in \{0,1,2,\ldots\}$, then $T_{m,l}(U) \subseteq V$, for all $m \in \{0,1,2,\ldots\}$.

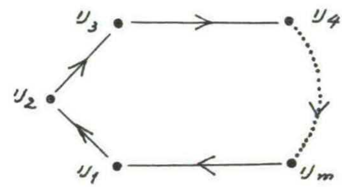
<u>Proof of lemma 4.3.16</u>

Take $l \in \{0,1,2,\ldots\}$ and assume (4.3.14), $T_{m,l}(U) \subseteq \hat{F}(V_n)$ for all $m \in \{0,1,2,\ldots\}$, there is a $R_X \in T_{t,l}(U)$, with $R_X \notin V$ and $t \in \{0,1,2,\ldots\}$.

It is sufficient to deduce a contradiction.

Without loss of generality suppose R_X is irreducible.

Hence, since R_X is a tournament there is a Hamilton circuit along R_X (See e.g. theorem 2.5.2). Let $\langle y_1,y_2,\ldots,y_t,y_1\rangle$ be such a circuit:



Since $R_X \in T_{t,l}(U)$, there is a relation $R'_X \in L(U)$, such that $\delta(R'_X,R_X) \le l$.

For some $i \in \{1,2,\ldots,t\}$ it holds that $\langle y_i,y_{i+1}\rangle \in R'_X$ (where $t + 1 \equiv 1$), otherwise R'_X is cyclic.

Without loss of generality let $\langle y_1,y_2\rangle \in R'_X$.

Since $R_X \in \hat{F}(V_n)$, there is a profile $r_X \in V$, with $\hat{F}(\tilde{r}) = R_X$.

Take $Y = X \cup \{a_1,a_2,a_3,\ldots,a_k\}$, such that $|Y| = t + k$.

Take $R''_Y \in L(U)$, such that

$\langle c,d\rangle \in R''_Y$ iff $\langle c,d\rangle \in R'_Y$, or
$\langle c,d\rangle = \langle a_i,a_j\rangle$ and $i \le j$, or
$\langle c,d\rangle = \langle y_1,a_i\rangle$ for some $i \in \{1,\ldots k\}$, or
$\langle c,d\rangle = \langle a_i,y_j\rangle$ and $\langle y_1,y_j\rangle \in R'_Y$ and $i \in \{1,\ldots k\}$, or
$\langle c,d\rangle = \langle y_j,a_i\rangle$ and $\langle y_j,y_1\rangle \in R'_Y$ and $i \in \{1,\ldots k\}$.

($R''_Y = \mathrm{Sub}(R'_X, y_1, \underset{\sim}{R}''_A)$, where $y_1a_1a_2a_3\ldots a_k : R''_A$).
Furthermore, take $R_Y \in T(U)$, such that
$\langle c,d\rangle \in R_Y$ iff $\langle c,d\rangle \in R_X$, or
$\langle c,d\rangle = \langle a_i,a_j\rangle$ and $i \le j$, or
$\langle c,d\rangle = \langle a_j,y_i\rangle$ and $\langle y_1,y_i\rangle \in R_X$, and $i \in \{1,\ldots k\}$, or
$\langle c,d\rangle = \langle y_i,a_j\rangle$ and $\langle y_i,y_1\rangle \in R_X$ and $i \in \{1,\ldots k\}$.

Then $\langle y_1a_1a_2a_3\ldots a_ky_2y_3\ldots y_t\rangle$ is a Hamilton circuit along R_Y. Hence, R_Y is irreducable. Furthermore it is obvious by the definition of R_Y, that $\delta(R_Y,R''_Y) = \delta(R_X,R'_X) = 1$.
Hence, $R_Y \in T_{t+k,1}(U) \subseteq F(V_n)$.
Now by lemma 4.3.13 it follows that $R_{Y|_X} \in V$.

This contradicts our assumptions.

■

We have just proved that $\cup\{T_{m,1}(U) : m \in \{1,2\ldots\}\}$ is not obtainable from V by a finite number of minimal extension, unless of course if it already is in V. The following lemma states that $\cup\{T_{m,l}(U) : m \in \{1,2,\ldots\}\}$ for $l \ge 1$ is in V.

Lemma 4.3.17

Assume (4.3.14).
Then for all $l \in \{0,1,2,\ldots\}$: if $T_{m,l}(U) \subseteq V$, for all $m \in \{0,1,2,\ldots\}$, then $T_{m,l+1}(U) \subseteq F(V_n)$, for all $m \in \{0,1,2,\ldots\}$.

Proof of lemma 4.3.17

Take $l \in \{0,1,2,\ldots\}$, assume (4.3.14), suppose $T_{m,l}(U) \subseteq V$, for all $m \in \{0,1,2,\ldots\}$, and $R_X \in T_{t,l+1}(U) - T_{t,l}(U)$, with $t \in \{0,1,2,\ldots\}$ and R_X is irreducible.
It is sufficient to prove that $R_X \in F(V_n)$.
Since $R_X \notin T_{t,l}(U)$ and $l \ge 0$ there is a $R'_X \in L(U)$ such that $\delta(R_X,R'_X) = l + 1 \ge 1$.
Hence, there are $y,x \in X$, such that $x>y : R_X$ and $y>x : R'_X$.
Hence, $\delta(R_{X}|_{X-\{x\}},R'_{X}|_{X-\{x\}}) \le l$ and $R_{X}|_{X-\{x\}} \in T_{t,l}(U) \subseteq V$.
Take $X_1 := \{b \in X: b > x : R_X\}$ and $X_2 := \{w \in X: w < x : R_X\}$.
Obviously $R_{X}|_{X_1} = (R_{X}|_{X-\{x\}})|_{X_1} \in T_{t,l}(U) \subseteq V$, unless $X_1 = \varphi$.

Similarly $R_{X|_{X_2}} \varepsilon V$, unless $X_2 = \varphi$.
Since R_X is irreducible, $X_1 \neq \varphi$ and $X_2 \neq \varphi$.
The following Condorcet-like profile r_X is in V:

$(R_{X|_{X_1}}) \text{ » } Id_{\{x\}} \text{ » } (R_{X|_{X_2}}) : R_X^i$ for all $i \varepsilon S \cap T$,

$Id_{\{x\}} \text{ » } (R_{X|_{X-\{x\}}}) \quad : R_X^i$ for all $i \varepsilon T \cap M$, and

$(R_{X|_{X-\{x\}}}) \text{ » } Id_{\{x\}} \quad : R_X^i$ for all $i \varepsilon M \cap S$.

Now take $x_1,y_1 \varepsilon X_1$, $x_2,y_2 \varepsilon X_2$ and $A = \{x,x_1,x_2,y_1,y_2\}$.
Let $a,b \varepsilon A$ and suppose $\langle a,b\rangle \varepsilon \bar{a}R_X$.
It suffices to show that $\{i \varepsilon N : a > b : R_X^i\} \varepsilon \{N,S,T,M\}$,
because of the quasi-decisiveness of these coalition.
The table below shows $\{i \varepsilon N : a > b : R_X^i\}$ for $\langle a,b\rangle \varepsilon \bar{a}R_X$.

\ b a\	x	x_1	x_2	y_1	y_2
x	\/ /\	\/ /\	T	\/ /\	T
x_1	S	\/ /\	N	N	N
x_2	\/ /\	M	\/ /\	M	N
y_1	S	N	N	\/ /\	N
y_2	\/ /\	M	N	M	\/ /\

Note that $x_1 = y_1$ or $y_2 = x_2$ does disturbe the reasoning presented here.

Hence, $\hat{F}(r_X) = R_X$.

■

<u>Proof of theorem 4.3.12</u>

Assume (4.3.14).
Note that $L(U) = T_{m,0}(U) \subseteq V$ for all $m \varepsilon \{1,2,...\}$
By a simple induction on l, and applying the lemmas 4.3.16 and 4.3.17 it follows that $T_{m,l}(U) \subseteq V$ for all m and l.

Hence, $V = T(U)$.

This contradicts our assumptions.

Hence, $\hat{F}$ is strongly dictatorial.

■

We conclude this section with some theorems about $\hat{F}(L_n(U))$, when $\hat{F}$ is an order morphism, non-dictatorial and $\hat{F}(L_n(U)) \subseteq T(U)$. Lemma 4.3.17 makes only use of the assumptions just stated. Since $\underset{\sim}{L}(U) \subseteq T_{m,0}(U)$ for all $m \in \{1,2,3...\}$ it follows that $T_{m,1}(U) \subseteq F(L_n(U))$. Having theorem 4.3.12 in mind the following question is natural: Is it deducible that $T(U) \subseteq \hat{F}(L_n(U))$? The answer to this question is negative, an argument is given in the following example.

Example 4.3.18

For every function $\hat{F} : L_n(U) \to \mathring{A}$, with $\hat{F}(L_n(A)) \subseteq \mathring{A}(A)$ for all $A \in Æ$, it holds that $T(U) \not\subseteq \hat{F}(L_n(U))$.

It is sufficient to prove that $|L_n(A)| < |T(A)|$ for some $A \in Æ$.

Note that if $|A| = p$, then $|L(A)_n^{\prime}| = (p!)^n$ and $|T(A)| = 2^{\frac{1}{2}p(p-1)}$.

Suppose $2^{k-1} < p \leq 2^k$.

Then $|L_n(A)| = (p!)^n \leq (p^p)^n = p^{np} \leq (2^k)^{n \cdot 2^k} = 2^{k \cdot n \cdot 2^k}$ and

$|T(A)| > 2^{\frac{1}{2}2^{k-1} \cdot (2^{k-1}-1)}$.

Since for fixed n but growing k $k \cdot n \cdot 2^k < \frac{1}{2} \cdot 2^{k-1} \cdot (2^{k-1}-1)$ it follows that $|L_n(A)| < |T(A)|$ for some large $|A|$.

Moreover, if we take $n = 3$, then for

- $|A| = 18$

$$
\begin{aligned}
|L_3(A)| = (18!)^3 &= (3^2 \cdot 2 \cdot 17 \cdot 2^4 \cdot 3 \cdot 5 \cdot 2 \cdot 7 \cdot 13 \cdot 2^2 \cdot 3 \cdot 11 \cdot 2 \cdot 5 \cdot \\
&\quad \cdot 3^2 \cdot 2^3 \cdot 7 \cdot 2 \cdot 3 \cdot 5 \cdot 2^2 \cdot 3 \cdot 2)^3 \\
&= 17^3 \cdot 13^3 \cdot 11^3 \cdot 7^6 \cdot 5^9 \cdot 3^{24} \cdot 2^{48} \\
&> 2^{12} \cdot 2^{21} \cdot 2^{16} \cdot 2^{20} \cdot 2^{36} \cdot 2^{48} = 2^{153} \\
&= 2^{9 \cdot 17} = 2^{\frac{1}{2} \cdot 18 \cdot (18-1)} \\
&= |T(A)|.
\end{aligned}
$$

- $|A| = 19$

  $|T(A)| = 2^{19 \cdot 9} = 2^{171}$ and

  $|L_3(A)| = 19^3 \cdot 17^3 \cdot 13^3 \cdot 11^3 \cdot 7^6 \cdot 5^9 \cdot 3^{24} \cdot 2^{48}$
  $< 2^{171}$.

So for n= 3, 19 is the smallest number of alternatives in A with $|L_n(A)| < |T(A)|$.

■

Since the answer of the question is negative it becomes even more interesting to know what $\hat{F}(L_n(U))$ looks like. It is pointed out here that we did not succeed in finding an operational description of $\hat{F}(L_n(U))$. On the other hand, we have some nice partial results, which will be discussed now. To avoid repetitions in the assumptions of the following theorems a general assumption is again formulated here.

<u>Assumption 4.3.19</u>

Let $n \geq 1$ be an integer and let $\hat{F}$ be an order morphism from $L_n(U)$ to $T(U)$, which is not strongly dictatorial.

■

Note that lemma 4.3.15 also holds when assumption 4.3.14 is substituted by assumption 4.3.19. The new lemma is referred to as lemma 4.3.15'. First an operational result is deduced.

<u>Lemma 4.3.20</u>

Assume (4.3.19), $R_Y \in T(U)$ and $R_{Y|_X} \in L(U)$, with

$Y = \{a,b,x_1,\dots,x_k\}$ and $X = \{x_1,x_2,\dots,x_k\}$.

Then $R_Y \in \hat{F}(L_n(U))$.

<u>Proof of lemma 4.3.20</u>

Assume the premisse of (4.3.20).

Without loss of genrality suppose $a > b : R_Y$.

Now there are four types of x-elements in X:

type 00, x_i ε X such that <x,a>,<x,b> ε R_Y,
a.——<——.x_i^{00} ——>——.b,
(To indicate the type we denote its corresponding number as subscript)

type 01, x_i ε X such that <x,a>,<b,x> ε R_Y,
a.——<——.x_i^{01} ——<——.b,

type 10, x_i ε X such that <a,x>,<x,b> ε R_Y,
a.——>——.x_i^{10} ——>——.b, and

type 11, x_i ε X such that <a,x>,<b,x> ε R_Y
a.——>——.x_i^{11} ——<——.b.

Note that the superscriptions correspond with the in- and outgoing arrows at x_i to or from a and b.
By theorem 4.3.6 it is sufficient to find a labeling, which satisfies (4.3.6.1) up to (4.3.6.4) with $\tilde{R}_Y$ in the rôle of R_A.
Take l : $\bar{n}$(Y x Y) -> 2^N as follows:
For <x,y> ε $\bar{n}R_Y$ let l(<x,y>) be defined according to the following tabel below:

\ y x\	a	b	01	00	10	11
a	\/ /\	S	\/ /\	\/ /\	S	S
b	\/ /\	\/ /\	M	\/ /\	\/ /\	S
01	T	\/ /\	N	S	S	S
00	T	T	M	N	S	S
10	\/ /\	T	M	M	N	S
11	\/ /\	\/ /\	M	M	M	S

, and

for <x,y> ε $\bar{v}\bar{n}R_Y$ (= $\bar{c}R_Y$) take l(<x,y>) := N - l(<x,y>), where S, T and M are the decisive coalition according to lemma 4.3.15'. Obviously (4.3.6.1), (4.3.6.2) and (4.3.6.3) hold for l. Note that in order to prove (4.3.6.4) for L it is sufficient to prove the following two:

(I) if $\begin{tikzcd} x \arrow[r] & y \arrow[r] & z \arrow[ll, bend left] \end{tikzcd}$: R_Y,

then $l(\langle x,y\rangle) \cap l(\langle y,z\rangle) \cap l(\langle z,x\rangle) = \varphi$, and

(II) if $\begin{tikzcd} x \arrow[r] \arrow[rr, bend right] & y \arrow[r] & z \end{tikzcd}$: R_Y,

then $l(\langle x,y\rangle) \subseteq l(\langle z,y\rangle)$ or $l(\langle z,x\rangle) \subseteq l(\langle z,y\rangle)$.

Now take the following matrix A :

$$\begin{bmatrix} \varphi & S & \varphi & \varphi & S & S \\ \varphi & \varphi & M & \varphi & \varphi & S \\ T & \varphi & N & S & S & S \\ T & T & M & N & S & S \\ \varphi & T & M & M & N & S \\ \varphi & \varphi & M & M & M & N \end{bmatrix}.$$

The $(i,j)^{th}$-cell of A indicates the individuals, who prefer the elements with type i, in the collection of {a,b,01,00,10,11}, to the elements with type j in the collection of {a,b,01,00,10,11}. Or stated otherwise if x has type i and y has type j, then the $(i,j)^{th}$-cell of A indicates the coalition which enables to go from x to y along R_Y. Furthermore $A \circ A$ can be calculated, where multiplication means intersecting and summation means uniting. The $(i,j)^{th}$-cell of $A \circ A$ is then the coalition which enables to go from x to y along R_Y by any intermidiate point $z \notin \{x,y\}$, where x has type i and y has type j.

Hence, in order to prove (I) and (II) it is sufficient to prove that all cells in $(A \circ A - A) \cap A^T$ are empty, where A^T is A transposed.

$A \circ A =$

$$\begin{bmatrix} \varphi & S \cap T & S \cap M & S \cap M & S & S \\ M \cap T & \varphi & M & S \cap M & S \cap M & S \\ T & S \cap T & N & S & S & S \\ T & T & M & N & S & S \\ M \cap T & T & M & M & N & S \\ M \cap T & M \cap T & M & M & M & N \end{bmatrix}$$

Hence, it is evident that (I) and (II) hold.

■

Now from (4.3.20) we can deduce the following result:

<u>Theorem 4.3.21</u>

Assume (4.3.19).

Then $T_5(U) \subseteq \hat{F}(L_n(U))$.

<u>Proof of theorem 4.3.21</u>

Let $R_X \in T_5(U)$ irreducible.

It is sufficient to prove that $R_X \in \hat{F}(L_n(U))$.

If $|X| \leq 4$, then $R_X \in T_{5,1}(U) \subseteq F(L_n(U))$.

If $|X| = 5$, then there is a triple $x,y,z \in X$ such that $R_X|_{\{x,y,z\}} \in L(U)$. Otherwise every triple is a 3-circuit of R_X, which cannot hold since the number of 3-circuits of R_X is less than the number of triples in X. Now we are done by lemma 4.3.20.

■

We have that $T_{m,0}(U) \subset T_{m,1}(U) \subseteq \hat{F}(L_n(U))$ for all m. A natural question therefore is $T_{m,2}(U) \subseteq \hat{F}(L_n(U))$? The answer to this question is positive.

<u>Theorem 4.3.22</u>

Assume (4.3.19).

Then $T_{m,3}(U) \subseteq \hat{F}(L_n(U))$ for all $m \in \{1,2,3,4..\}$.

<u>Proof of theorem 4.3.22</u>

Let $R_X \in T_{m,3}(U)$ be irreducible.

It is sufficient to prove that $R_X \in \hat{F}(L_n(U))$.

Since $R_X \in T_{m,3}(U)$ is irreducible there is a relation $R'_X \in L(U)$ such that $\delta(R_X,R'_X) \leq 3$.

Let $x_1x_2x_3...x_m : R'_X$.

Now because $\delta(R_X,R'_X) \leq 3$ there are i_1,j_1,i_2,j_2,i_3 and j_3 in $\{1,2,3,...m\}$ such that $i_1 + 1 < j_1$, $i_2 + 1 < j_2$, $i_3 + 1 < j_3$ and $R_X = \langle(R'_X - \{\langle x_{i_1},x_{j_1}\rangle, \langle x_{i_2},x_{j_2}\rangle, \langle x_{i_3},x_{j_3}\rangle\}) \cup \{\langle x_{j_1},x_{i_1}\rangle, \langle x_{j_2},x_{i_2}\rangle, \langle x_{j_3},x_{i_3}\rangle\}),X\rangle$.

$\langle x_{j_k}, x_{i_k}\rangle$ for $k \in \{1,2,3\}$ are the arrows which should be reversed to obtain R'_X from R_X.

Without loss of generality suppose $i_1 \le i_2 \le i_3$.

Since R_X is irreducible $i_1 = 1$.

Now if $|\{i_1, i_2, i_3, j_1, j_2, j_3\}| \le 5$, then there are $i,j \in \{i_1, i_2, i_3, j_1, j_2, j_3\}$ such that $i = j$. Obviously $R_X|_{X - \{x_i\}} \in T_{m,1}(U)$. Hence, there is a $Y \subseteq X$ such that $|Y| = |X| - 2$ and $R_X|_Y \in L(U)$. But then we are done by lemma 4.3.20.

Suppose $|\{i_1, i_2, i_3, j_1, j_2, j_3\}| = 6$.

Then we are done by theorem 4.3.23, which we will prove hereafter.

■

In theorem 4.3.22 it is proven that $\cup\{T_{m,3}(U) : m \in \{1,2..\}\}$ is contained in $\hat{F}(L_n(U))$, unless $\hat{F}$ is dictatorial. It is clear that $\cup\{T_{m,3}(U) : m \in \{1,2..\}\}$ is an odd set, that is its basic elements, by which all the other elements can be constructed using permutation, concatenation, convertation, restriction and substitution as construction tools, are enumerous.

The following theorem, the last of this section, shows that the basic elements of $\hat{F}(L_n(U))$ cannot be described like those of $\cup\{T_{m,3}(U) : m \in \{1,2..\}\}$. And by this theorem it becomes questionable if it is possible at all to describe all those basic elements of $\hat{F}(L_n(U))$. Hence, we are settled with the problem: What looks the set $\hat{F}(L_n(U))$ like?

First we introduce a new set of orderings.

$T_?(U) := \{R_X \in T(U)$: there is a $R'_X \in L(U)$, an integer $k \ge 0$, a set $\{i_1,i_2,\ldots,i_k,j_1,j_2,\ldots,j_k\}$, with $|\{i_1,i_2,\ldots,i_k,j_1,j_2,\ldots,j_k\}| = 2k$, and a set $\{x_1,x_2,\ldots,x_p\} = X$, such that $x_1x_2\ldots x_k : R'_X$ and

$$R_X = (R'_X \cup \langle\{\langle x_{j_t},x_{i_t}\rangle : t \in \{1,\ldots,k\}\},X\rangle) - \langle\{\langle x_{i_t},x_{j_t}\rangle : t \in \{1,\ldots,k\}\},X\rangle \quad \}.$$

$T_?(U)$ is the set of tournaments, which can be obtained form a linear ordering by switching around a finite number of different and disjoint pairs of that ordering. It is, although cumbersome, straightforward to prove that $T_?(U)$ can be classified as a set of reflexive, antisymmetric and complete orderings. Hence, $T_?(U) \subseteq T(U)$.

On the other hand, it is not easy, if at all, to describe $T_?(U)$. It is not evident to decide whether an arbitrary chosen tournament R_X is in $T_?(U)$ or not. Therefore the set is subscribed with a questionmark. Of course it is possible to check for each linear ordering $R'_X \in L(U)$, whether or not R_X and R'_X satisfy the formulation. But this procedure can cost a lot of checkings (Note that $|L(X)| = |X|!$). Having these arguments in mind, by the following theorem, it is clear that $\hat{F}(L_n(U))$ is not easily described.

Theorem 4.3.23

Assume (4.3.19). Then $T_?(U) \subseteq \hat{F}(U)$.

Proof of theorem 4.3.23

By lemma 4.3.15' there are S, T and M in 2^N, such that $qD(\hat{F},S) = qD(\hat{F},T) = qD(\hat{F},M) = U2$.

It is sufficient to prove that:

For all $k \geq 0$ and $R_X \in T(U)$, such that $R'_X \in L(U)$, $x_1x_2\ldots x_k : R'_X$, $\{i_1,i_2,\ldots,i_k,j_1,j_2,\ldots,j_k\} \subseteq \{1,2,\ldots,p\}$, $|\{i_1,i_2,\ldots,i_k,j_1,j_2,\ldots,j_k\}| = 2\cdot k$ and

$$R_X = (R'_X \cup \langle\{\langle x_{j_t},x_{i_t}\rangle : t \in \{1,\ldots,k\}\},X\rangle) - \langle\{\langle x_{i_t},x_{j_t}\rangle : t \in \{1,\ldots,k\}\},X\rangle$$

there is a $r_X \in L_n(U)$, such that $R'_X = R^i_X$ for all $i \in S \cap T$, and $\hat{F}(r_X) = R_X$.

The proof is by induction on k.

Basis: $k = 0$ is trivial, since $R_X \in L(U)$.

By the Pareto-optimality $\hat{F}(\langle R_X,\ldots,R_X\rangle) = R_X$.

Induction step: Suppose it holds for numbers less or equal to $k - 1$ and $\tilde{R}_X = (R_X - \langle\{\langle x_{j_1},x_{i_1}\rangle\},X\rangle) \cup \langle\{\langle x_{i_1},x_{j_1}\rangle\},X\rangle$.

Then by the induction hypothesis there is a profile $\tilde{r}_X \in L_n(U)$, with $R'_X = \tilde{R}^i_X$ for all $i \in S \cap T$ and $\hat{F}(\tilde{r}_X) = \tilde{R}_X$.

Take $r_X \in L_n(U)$ as follows:

$R^i_X = R'_X$ for all $i \in S \cap T$,

$R_X^i = \tilde{R}_X^i \gg R'_X|_{\{x_{j_1}\}} \gg R'_X|_{\{x_{i_1}\}}$ for all $i \in S \cap M$, and

$R_X^i = R'_X|_{\{x_{j_1}\}} \gg R'_X|_{\{x_{i_1}\}} \gg \tilde{R}_X^i$ for all $i \in T \cap M$.

Since $\tilde{r}_X|_{X-\{x_{i_1},x_{j_1}\}} = r_X|_{X-\{x_{i_1},x_{j_1}\}}$ it follows that

$\tilde{R}_X|_{X-\{x_{i_1},x_{j_1}\}} = \hat{F}(r_X)|_{X-\{x_{i_1},x_{j_1}\}}$.

By the decisiveness of M it follows that $\langle x_{j_1},x_{i_1}\rangle \in \bar{a}\hat{F}(r_X)$.
If $x_t > x_{i_1} : R_X$ and $t \neq j_1$, then by the decisiveness of S
$\langle x_t,x_{i_1}\rangle \in \bar{a}\hat{F}(r_X)$.
Similarly $\langle x_t,x_{j_1}\rangle \in \bar{a}\hat{F}(r_X)$, if $\langle x_t,x_{j_1}\rangle \in R_X$ and $t \neq i_1$.
If $x_{i_1} > x_t : R_X$ and $t \neq j_1$, then by the decisiveness of T
it follows that $\langle x_{i_1},x_t\rangle \in \bar{a}\hat{F}(r_X)$.
Similar $\langle x_{j_1},x_t\rangle \in \bar{a}\hat{F}(r_X)$, if $\langle x_{j_1},x_t\rangle \in R_X$ and $t \neq i_1$.
Hence, $\hat{F}(r_X) = R_X$ and we are done.

■

We have just seen the set $\hat{F}(L_n(U))$ contains the set $T_?(U)$. Looking back to theorem 4.3.8, 4.3.12, 4.3.22 and 4.3.23 one may come to the conclusion that the price we have to pay in order to avoid dicatorship in an order morphism, is that its range is not easy to opeate within the classification system. For cases where this operation is described in transitivity terms, it is theorem 4.3.8 which states that the range is not easy to operate. For cases, where this operation is described in terms of finite constructions with tools as permutation, concatenation, substitution, conversion, and restriction, theorems 4.3.12 and 4.3.23 state that the range is not easy to operate.

In theorem 4.2.8 an impossibility theorem for choice correspondences is proved by virtue of a correspondence between by orderings reconstructable choice correspondences and order morphisms. Although, this is not new in literature (See e.g. Gibbard [1973], Satterthwaite [1975] and Muller & Satterthwaite [1977], Satterthwaite [1975] even implicitly defined rationalizability), the approach followed in § 4.2 is more general (decision procedures others than those choosing the best elements are <u>not</u> excluded). We therefore expect that new impossibilities for choice correspondences can be derived from, e.g., theorem 4.3.23. For reasons of time and space this is not done here.

§ 4.4 Impossibility Theorems Based on Non-expansiveness

In this section the effect of a weakening of the independence of irrelevant alternatives condition is investigated by a kind of continuity condition, called non-expansiveness. It will give a dieper insight in the significance of the independence of irrelevant alternatives for impossibility results, because it does not lead to a possible construction of a non-trivial welfare function. Since we weaken the independence of irrelevant alternatives condition, we cannot substitute anymore an order morphism for a welfare function. Therefore we will return to the original model as introduced in section 1.1. Furthermore, the neutrality condition will be dropped. The range of the welfare functions studied here is restricted only to the set of quasi-orderings, Q(A), for a finite set of alternatives A. (See Storcken [1989] for similar work but with range of A(A) and neutrality). We will prove that there does not exist a non-dictatorial, non-expansive and Pareto-optimal welfare function from $L_n(A)$ to Q(A).

Firstly some distance functions on sets of orderings are introduced and compared with similar distance functions proposed in literature. Then it is shown that independence of irrelevant alternatives implies non-expansiveness of a welfare function, but not the other way around. Finally several impossiblity theorems are proved using non-expansiveness.

In this section it will be assumed that the set of alternatives A is fixed. To simplify notations we therefore skip the subscript "A" at every relation or profile in this entire section.

Let us start with distance functions on the set of quasi-orderings. In literature the following distance function on orderings in relation with social choice functions is often used: $\hat{d}(R^1_A,R^2_A) = |(R^1_A - R^2_A) \cup (R^2_A - R^1_A)|$ (See e.g. Kemeny & Snell [1962], Bogart [1973], Monjardet [1981], Barthélemy & Flament & Monjardet [1982], Bezembinder [1981] and Bogart [1982]). We will introduce here other distance functions. Non-expansiveness based on our distance functions, is implied by the independence of irrelevant alternatives. The non-expansiveness based on $\hat{d}$ is a very restrictive condition for welfare functions; this is demonstrated in example 4.4.1.

<u>Example 4.4.1.</u>

Suppose $\Gamma = \langle A,N\rangle$ is a society, with $A = \{a,b,c\}$ and $N = \{1,2,3,...,n\}$, and F is a strongly Pareto-optimal welfare function on Γ from $W_n(A)$ to $W(A)$.
Define $\hat{d}_n : W_n(A) \times W_n(A) \to [0,\infty\rangle$, for all $r,\hat{r} \in W_n(A)$ as follows: $\hat{d}_n(r,\hat{r}) := \Sigma_{i=1}^{n} \hat{d}(R^i,\hat{R}^i)$. $\hat{d}_n$ is a distance function.
Then F is not $\langle \hat{d}_n,\hat{d}\rangle$-non-expansive.
Suppose F is $\langle \hat{d}_n,\hat{d}\rangle$-non-expansive.
Take the following profiles in $W_n(A)$.

$\bar{r}$: abc : $\bar{R}^1$,	$\tilde{r}$: (ab)c : $\tilde{R}^1$,	$\hat{r}$: abc : $\hat{R}^1$,
cba : $\bar{R}^2$, and	cba : $\tilde{R}^2$, and	c(ba) : $\hat{R}^2$, and
(abc) : $\bar{R}^i$,	(abc) : $\tilde{R}^i$,	(abc) : $\hat{R}^i$,
for $i > 2$,	for $i > 2$,	for $i > 2$.

Then $\hat{d}_n(\bar{r},\tilde{r}) = \hat{d}(\bar{R}^1,\tilde{R}^1) = 1$ and $\hat{d}_n(\bar{r},\hat{r}) = \hat{d}(\bar{R}^2,\hat{R}^2) = 1$.
Hence, $\hat{d}(F(\bar{r}),F(\tilde{r})) \le 1$ and $\hat{d}(F(\bar{r}),F(\hat{r})) \le 1$. (4.4.1.1.)
Note furthermore that $a > b : F(\hat{r})$ and $b > a : F(\tilde{r})$ since F is strongly Pareto-optimal.
Hence, $a \sim b : F(\bar{r})$, otherwise (4.4.1.1) is not satisfied.
Similarly it follows that $b \sim c : F(\bar{r})$. Since $F(\bar{r}) \in W(A)$ we have $(abc) : F(\bar{r})$. By (4.4.1.1) it then follows that: $(bc) : F(\tilde{r})$. Hence, $(bc)a : F(\tilde{r})$, since $F(\tilde{r}) \in W(A)$. But then $\hat{d}(F(\tilde{r}),F(\bar{r})) > 1$, which contradicts (4.4.1.1).

■

In the previous example we have just seen that the strongly Pareto-optimality and the $\langle \hat{d}_n,\hat{d}\rangle$-non-expansiveness exclude each other, when we consider welfare functions from $W_n(A)$ to $W(A)$. A fact which is not true for the conditions strongly Pareto-optimality and the independence of irrelevant alternatives: Dictatorial welfare function can be strongly Pareto-optimal and independent of irrelevant alternatives. By these facts it is clear that $\langle \hat{d}_n,d_n\rangle$-non-expansiveness is not a good substitute for the independence condition. Therefore other distance functions will be introduced.

The distance functions introduced here are based on the concept of elementary changes. Moreover, these elementary changes play an important rôle in the impossibility theorems deduced later on. The following notion is used in the explanation of elementary change.

<u>Definition 4.4.2</u> <u>Succession</u>

Let x,y be two different elements in A and let R be a relation on A. x and y <u>succeed each other in</u> R, iff there are no $z_1,z_2,\ldots z_k \in A - \{x,y\}$ such that $\langle x,z_1\rangle$, $\langle z_1,z_2\rangle,\ldots$ $\langle z_{k-1},z_k\rangle$, $\langle z_k,y\rangle \in \bar{a}R$, or $\langle y,z_1\rangle$, $\langle z_1,z_2\rangle,\ldots,$ $\langle z_{k-1},z_k\rangle$, $\langle z_k,x\rangle \in \bar{a}R$.

Notation: $x \sim y$: sucR, where sucR is the set of all pairs which succeed each other in R.

■

If x succeeds y in R, then there is no path along $\bar{a}R$ going from x to y or from y to x, which visits another element $z \in A - \{x,y\}$. For instance suppose $R := \{\langle x,y\rangle,\langle y,z\rangle,\langle z,t\rangle\}$. Then $x \sim y$: sucR, $y \sim z$: sucR, $z \sim t$: sucR, but not: $x \sim t$: sucR. If furthermore, $A = \{x,y,z,t,s\}$ then $x \sim s$: sucR, $y \sim s$: sucR, $z \sim s$: sucR and $t \sim s$: sucR. The relaton "succeed" is symmetric, irreflexive and neither complete nor transitive in general.

Although the notion of elementary changes can be introduced appart from the notion of succesion, it is difficult to explain the former without the last. Now the definition of elementary changes is stated.

<u>Definition 4.4.3</u> <u>Elementary Change</u>

Suppose $V_n(A)$ is a classified set of profiles on A, r^1, $r^2 \in V_n(A)$ and $x,y \in A$.

4.4.3.1 r and $\hat{r}$ <u>form an elementary change of each other in</u> $\{x,y\}$, iff $R^i \cap E_{xy} = \hat{R}^i \cap E_{xy}$, for all $i \in \{1,2,3,\ldots,n\}$.

Where $E_{xy} := \bar{c}(\{x,y\} \times \{x,y\})$.

Notation: $EC(V_n(A),\{x,y\}) = \{\langle r,\hat{r}\rangle : r,\hat{r} \in V_n(A)$ form an elementary change of each other in $\{x,y\}\}$.

4.4.3.2 r and $\hat{r}$ <u>form an elementary change of each other</u>, iff there are $a,b \in A$, such that r and $\hat{r}$ form an elementary change of each other in $\{a,b\}$.

Notation: $EC(V_n(A)) := \cup\{EC(V_n(A),\{x,y\}) : x,y \in A\}$.

■

r and $\hat{r}$ in $V_n(A)$ form an elementary change of each other in $\{x,y\}$, iff each of the corresponding components of r and $\hat{r}$ only differs on the pair x and y. $EC(V_n(A),\{x,y\})$ is an equivalence relation on $V_n(A)$ and $EC(V_n(A))$ is a reflexive and symmetric relation on $V_n(A)$. The notion of elementary change has already been introduced (See e.g. Moulin [1983]).

The next theorem provides us with tools to understand the formal notion.

<u>Theorem 4.4.4</u>

Suppose $a,b,x,y \varepsilon A$ and $r,\hat{r} \varepsilon Q_n(A)$.

4.4.4.1 If $\langle r,\hat{r}\rangle \varepsilon EC(Q_n(A),\{x,y\})$, then for all $i \varepsilon \{1,\ldots,n\}$: either $R^i = \hat{R}^i$, or ($x \sim y$: $sucR^i$ and $x \sim y$: $suc\hat{R}^i$).

4.4.4.2 $EC(Q_n(A),\{x,y\}) = Id_{Q_n(A)}$, iff $x = y$.

4.4.4.3 $EC(Q_n(A),\{a,b\}) \cap EC(Q_n(A),\{x,y\}) \neq Id_{Q_n(A)}$, iff $\{x,y\} = \{a,b\}$ and $x \neq y$.

<u>Proof of theorem 4.4.4</u>

(4.4.4.1) Let $r,\hat{r} \varepsilon Q_n(A)$ such that $\langle r,\hat{r}\rangle \varepsilon EC(Q_n(A),\{x,y\})$.

Furthermore, suppose $R^i \neq \hat{R}^i$ and not $x \sim y$: $sucR^i$.

It suffices to deduce a contradiction.

The following holds without loss of generality:

$R^i \cap E_{xy} = \hat{R}^i \cap E_{xy}$ and $\langle x,z_1\rangle$, $\langle z_1,z_2\rangle,\ldots$ $\langle z_2,z_k\rangle$, $\langle z_k,y\rangle \varepsilon \bar{a}R^i$, where $z_1,\ldots z_k \varepsilon A - \{x,y\}$.

Hence, $\langle x,y\rangle \varepsilon \bar{a}R^i$, since $\hat{R}^i \varepsilon Q(A)$ and $\langle x,z_1\rangle$, $\langle z_1,z_2\rangle,\ldots,$ $\langle z_k,y\rangle \varepsilon \bar{a}\hat{R}^i$. Hence, $R^i = \hat{R}^i$, which contradicts our assumptions.

(4.4.4.2) Evident.

(4.4.4.3) (if) is trivial by (4.4.4.2).

(only if) Suppose $\{x,y\} \neq \{a,b\}$.

If either $x = y$ or $a = b$ we are again done by (4.4.4.2).

Suppose furthermore $a \neq b$ and $x \neq y$.

Let $\langle r,\hat{r}\rangle \varepsilon EC(Q_n(A),\{x,y\}) \cap EC(Q_n(A),\{a,b\})$. It suffices to prove that $r = \hat{r}$. But this is obvious since for all $i \varepsilon \{1,2,\ldots,n\}$: $R^i \cap E_{xy} = \hat{R}^i \cap E_{xy}$ and $R^i \cap E_{ab} = \hat{R}^i \cap E_{ab}$, where $E_{ab} \cup E_{xy} = A \times A$.

■

Note that (4.4.4) gives us further information about an elementary change. Two relations R^1 and R^2 in $Q(A)$ form an elementary change in $\{x,y\}$, iff $R^1 = R^2$ or x and y succeed each other in R^1 and R^2 and they differ only on x and y.

In the rest of this section $EC(Q(A))$ and $EC(L_n(A))$ play an important rôle. Therefore some pictorial information about these relations is given. That is the neighbourhood graphs of distance functions. First these distance functions are introduced.

<u>Definition 4.4.5</u> $\tilde{q}_n$ <u>distance based on elementary changes on</u> $Q_n(A)$

$\tilde{q}_n$ is a function from $Q_n(A) \times Q_n(A)$ to $\{0,1,2,3,4...\}$, such that for all $r,\hat{r} \in Q_n(A)$:

$$\tilde{q}_n(r,\hat{r}) := \min \{k : \text{there are } r^0=r, r^1, r^2, ..r^k=\hat{r} \in Q_n(A), \text{ such that for all } i \in \{0,1,...,k-1\}: \langle r^i, r^{i+1}\rangle \in EC(Q_n(A))\}$$

■

$\tilde{q}_n(r,\hat{r})$ is the minimum number of elementary changes needed to transform r into $\hat{r}$, such that all these changes result in intermediate profiles which are still in $Q_n(A)$. Before continuing it is inevitable to proof:

<u>Theorem 4.4.6</u>

$\tilde{q}_n$ is a well-defined distance function on $Q_n(A)$.

<u>Proof of theorem 4.4.6</u>

It is sufficient to proof that for all $R \in Q(A)$ there are $R^0 = R$, R^1, $R^2 ... R^k = A \times A$, such that $\langle R^i, R^{i+1}\rangle \in EC(Q(A))$ for all $i \in \{0,1,...,k-1\}$.

Let R be in $Q(A)$. Because $|A|$ is finite there is a pair $x,y \in A$ such that $x \sim y$: sucR. If for all those pairs $x \sim y : R$ we are done since then $\bar{a}R = \varphi$. If $x > y : R$, take $R' = R \cup \{\langle y,x\rangle\}$. Obviously $\langle R,R'\rangle \in EC(Q(A))$ and by a simple induction reasoning on $|\bar{a}R|$ we are done.

■

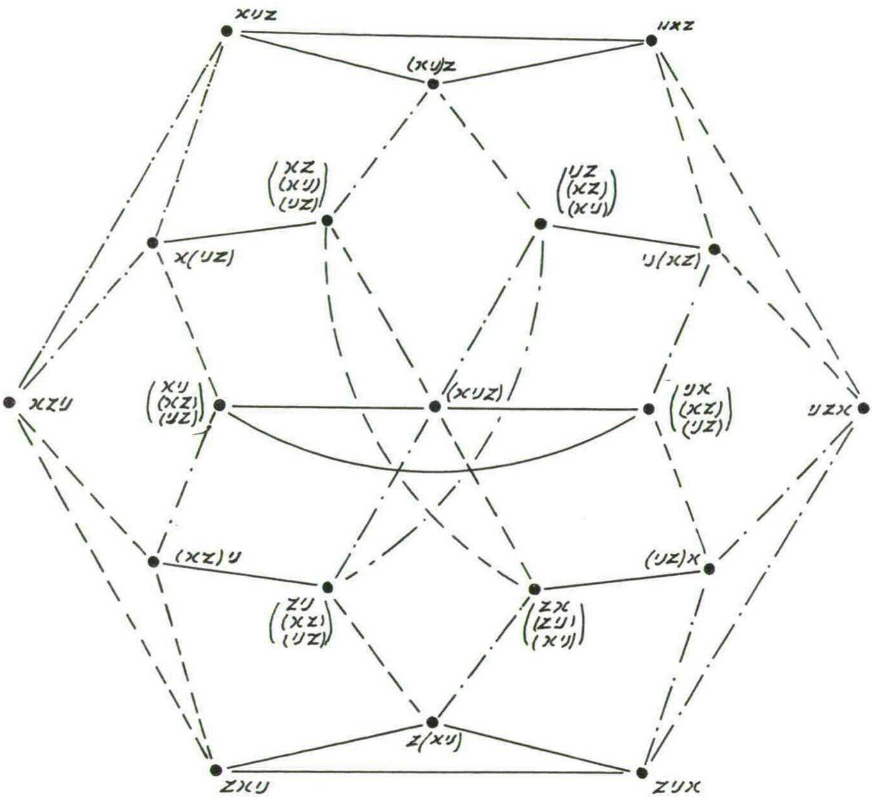
To avoid indices, as notational convention we will write $\tilde{q}$ instead of $\tilde{q}_1$.

The neighbourhood graph $G_{\langle Q(A),\tilde{q}\rangle}$ is obiously a graph which visualizes $EC(Q(A))$. Let us draw this graph for $|A| = 3$, because it plays an important rôle in the impossibility theorems discussed here after.

Example 4.4.7

Let A = {x,y,z}.

By the foregoing it is straightforward that the neighbourhood graph of $\langle Q(A),\underset{\sim}{q}\rangle$ is as follows:



Here the 19 relations in Q(A) are represented by their representation how they order x,y and z. For instance xyz : R means R = $\bar{r}${<x,y>,<x,z>,<y,z>} or graphically:

```
x.——>——.y——>——.z : R.
└———————>———————┘
```

(xy)z : R' means R' = R U {<y,x>} or graphically:

```
x.—————.y——>——.z : R'.
└———————>———————┘
```

$\begin{bmatrix} yz \\ (xy) \\ (xz) \end{bmatrix}$: R" means R" = R' U {<z,x>} or graphically:

x.———.y——>——.z : R".

The other relations have similar interpretations.

Furthermore, if R^1 ——— R^2, then $<R',R^2> \varepsilon$ EC(Q(A),{x,y}),

if R^1 —·— R^2, then $<R',R^2> \varepsilon$ EC(Q(A),{y,z}),

and if R^1 – – – R^2, then $<R',R^2> \varepsilon$ EC(Q(A),{x,z}).

Naturally there are only $\binom{3}{2} = 3$ types of elementary changes.

■

Notice that $<Q_n(A),\tilde{q}_n>$ is a full metric space, which evidently follows from the definition of the $\tilde{q}_n$-distance. $\tilde{q}_n$ is defined on the basis of elementary changes in $Q_n(A)$. Of course we can also consider elementary changes in $L_n(A)$ in that case we get:

<u>Definition 4.4.8</u> $\tilde{l}_n$ <u>-distance based on elementary changes on</u> $L_n(A)$

$\tilde{l}_n$ is a function from $L_n(A) \times L_n(A)$ to {0,1,2,...}, such that for all $r,\hat{r} \varepsilon L_n(A)$:

$\tilde{l}_n(r,\hat{r})$:= min{k : there are $r^0 = r, r^1, \ldots r^{k-1}, r^k = \hat{r} \; \varepsilon \; L_n(A)$, such that for all i ε {0,1,2,...k-1}: $<r^i,r^{i+1}> \varepsilon \; EC(L_n(A))$}.

■

$\tilde{l}_n$ has a similar interpretation as $\tilde{q}_n$, with the only difference that the elementary steps are now restricted to $L_n(A)$. Formally it should be proven that $\tilde{l}_n$ is a distance function. Again it is sufficient to prove that $EC(L_n(A))$ is connected. But since this is straightforward to prove, we have:

<u>Theorem 4.4.9</u>

$\tilde{l}_n$ is a well-defined distance function on $L_n(A)$.

■

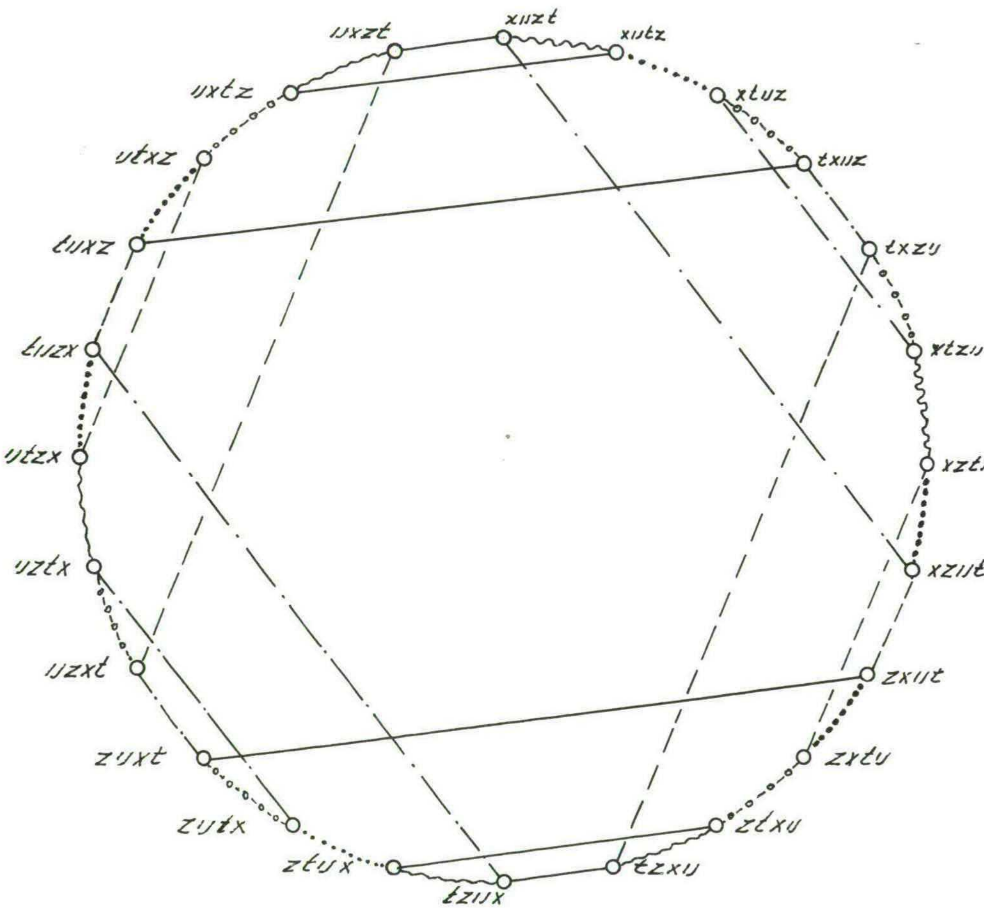
Evidently for all $r,\hat{r} \; \varepsilon \; L_n(A)$ it holds that $\tilde{l}_n(r,\hat{r}) \geq \tilde{q}_n(r,\hat{r})$. In general we do not know, whether or not this inequality is strict.

Again by definition $<L_n(A),\tilde{l}_n>$ is a full metric space and as notational convention $\tilde{l}$ is used instead of $\tilde{l}_1$.

Next the neighbourhood graph of $\langle L_n(A),\tilde{l}\rangle$ is drawn for $|A| = 4$.

<u>Example 4.4.10</u>

Take $A = \{x,y,z,t\}$. It is cumbersome but straightforward to prove that the neighbourhood graph of $\langle L_n(A),\tilde{l}\rangle$ is as follows:



Here again the relations are represented by their representation of the ordering of the elements in A.

Furthermore: if R^1 ——— R^2, then $\langle R^1,R^2\rangle \in EC(L(A),\{x,y\})$,
if R^1 ·—·— R^2, then $\langle R^1,R^2\rangle \in EC(L(A),\{y,z\})$,
if R^1 -——— R^2, then $\langle R^1,R^2\rangle \in EC(L(A),\{x,z\})$,
if R^1 .... R^2, then $\langle R^1,R^2\rangle \in EC(L(A),\{y,t\})$,
if R^1 ∿∿∿ R^2, then $\langle R^1,R^2\rangle \in EC(L(A),\{z,t\})$,
and if R^1 -o-o-o- R^2, then $\langle R^1,R^2\rangle \in EC(L(A),\{x,t\})$.

■

After introducing $\tilde{q}_n$ and $\tilde{l}_n$ distances it is a small step to think of an introduction of a distance function based on elementary steps for any classified set of orderings. But taking the weak orderings for instance it follows from example 4.4.7 that there is no elementary change from the relation (xyz) to any other weak ordering. Hence, $EC(W_n(A))$ is not connected in general and therefore $\langle W_n(A),\tilde{w}_n\rangle$ is not a metric space. So we have to be very careful, when defining a distance function based on elementary changes in a classified set of profiles.

In this section only one further distance function will be discussed.

<u>Definition 4.4.11</u> $\tilde{c}_n$-<u>distance based on elementary changes on</u> $C_n(A)$.

$\tilde{c}_n$ is a function from $C_n(A) \times C_n(A)$ to $\{0,1,2,...\}$, such that for all $r,\hat{r} \in C_n(A)$:

$$\tilde{c}_n(r,\hat{r}) := \min \{k : \text{There are } r^0=r,r^1,r^2,\ldots r^k=\hat{r} \in C_n(A), \text{ such that for all } i \in \{0,1,2,\ldots k-1\}: \langle r^i,r^{i+1}\rangle \in EC(C_n(A))\}.$$

■

Again $\tilde{c}_n$ can be interpretated similar to $\tilde{q}_n$ and $\tilde{l}_n$. Moreover, $\tilde{c}_n$ is a distance function, since $EC(C_n(A))$ is connected. Again it follows that $\langle C_n(A),\tilde{c}_n\rangle$ is a full metric space. Furthermore, as a notational convention $\tilde{c}$ is written instead of $\tilde{c}_1$. Before we start with a comparison of the different distance functions introduced here let us make a remark on the distances compared with the $\hat{d}_n$-distance. The main difference between $\tilde{c}_n$, $\tilde{q}_n$ and $\tilde{l}_n$ on one side and $\hat{d}_n$ on the other is the difference in the betweenness relation. For instance (xy)z is between $\underset{\sim}{xyz}$ and $\underset{\sim}{yxz}$ according to the $\hat{d}_1$-distance. This is not the case for c, q and l. Noting that this betweenness property was extensively used in example 4.4.1 it becomes hopeful that $\tilde{c}_n$, $\tilde{q}_n$ and $\tilde{l}_n$ are suitable distances. That is, perhaps they do not have the draw-back of conflicting with the strong Pareto-optimality.

Let us now come to some properties of these distance functions.

<u>Theorem 4.4.12</u>

Suppose $A \in Æ$, $N = \{1,2,\ldots,n\}$ and $r,\hat{r} \in C_n(A)$.

Then $\tilde{c}_n(r,\hat{r}) = \frac{1}{2}|\cup\{R^i \triangle \hat{R}^i : i \in N\} \cup \cup\{\bar{v}R^i \triangle \bar{v}\hat{R}^i : i \in N\}|$, where $R^i \triangle \hat{R}^i := (R^i - \hat{R}^i) \cup (\hat{R}^i - R^i)$ is the symmetric difference between R^i and $\hat{R}^i$.

<u>Proof of theorem 4.4.12</u>

Note that: $\langle x,y\rangle \in R^i \triangle \hat{R}^i$, iff $\langle y,x\rangle \in \bar{v}R^i \triangle \bar{v}\hat{R}^i$.
Hence, $\frac{1}{2}|(R^i \triangle \hat{R}^i) \cup (\bar{v}R^i \triangle \bar{v}\hat{R}^i)|$ is equal to the number of not ordered pairs $\{x,y\}$ on which R^i and $\hat{R}^i$ differ. Hence, $\frac{1}{2}|\cup\{R^i \triangle \hat{R}^i : i \in N\} \cup \cup\{\bar{v}R^i \triangle \bar{v}\hat{R}^i : i \in N\}|$ is equal to the number of not ordered pairs on which r and $\hat{r}$ differ. So $\tilde{c}_n(r,\hat{r}) \geq \frac{1}{2}|\cup\{R^i \triangle \hat{R}^i : i \in N\} \cup \cup\{\bar{v}R^i \triangle \bar{v}\hat{R}^i : i \in N\}|$.
Since $R = (R' \cap E_{xy}) \cup (R'' \cap \{x,y\} \times \{x,y\})$ is strongly complete if R' and R" are strongly complete it follows that all these differences between r and $\hat{r}$ cause only as many elementary steps as there are different ordered pairs involved. Hence,

$$\tilde{c}_n(r,\hat{r}) \leq \frac{1}{2}|\cup\{R^i \triangle \hat{R}^i : i \in N\} \cup \cup\{\bar{v}R^i \triangle \bar{v}\hat{R}^i : i \in N\}|.$$

■

The following theorem uncovers some logical relationship between the distances introduced here.

<u>Theorem 4.4.13</u>

Let $A \in Æ$.

4.4.13.1 For all $\underset{\sim}{r},\hat{\underset{\sim}{r}} \in Q_n(A)$: $\tilde{\underset{\sim}{c}}_n(r,\hat{r}) \leq \tilde{\underset{\sim}{q}}_n(r,\hat{r})$.

4.4.13.2 For all $\underset{\sim}{r},\hat{\underset{\sim}{r}} \in L_n(A)$: $\tilde{\underset{\sim}{c}}_n(r,\hat{r}) \leq \tilde{\underset{\sim}{q}}_n(r,\hat{r}) \leq \tilde{l}_n(r,\hat{r})$.

4.4.13.3 For all $R,\hat{R} \in L(A)$: $c(R,\hat{R}) = q(R,\hat{R}) = l(R,\hat{R})$.

Proof of theorem 4.4.13

(4.4.13.1) and (4.4.13.2) are evident.

(4.4.13.3) It is sufficient to prove by induction on $k \geq 0$, that for all $R, \hat{R} \in L(A)$, with $\tilde{c}(R,\hat{R}) \leq k$:

$\tilde{l}(R,\hat{R}) = \frac{1}{2}|(R \triangle \hat{R}) \cup (\bar{v}R \triangle \bar{v}\hat{R})|$.

Basis: $k = 0$ is trivial.

Induction step: Suppose $R, \hat{R} \in L(A)$ are such that $\tilde{c}(R,\hat{R}) = k$.

Obviously since $A = \{a_1, a_2, \ldots, a_p\}$ is finite there are a_i and a_{i+1} such that $a_1a_2a_3\ldots a_ia_{i+1}\ldots a_p : R$ and $a_i < a_{i+1} : \hat{R}$.

Take $a_1a_2a_3\ldots a_{i-1}a_{i+1}a_ia_{i+2}\ldots a_p : R'$ in $L(A)$.

Then $\langle R,R'\rangle \in EC(L(A),\{a_i,a_{i+1}\})$ and $\tilde{c}(R,R') = 1$.

Moreover it is evident that $\tilde{c}(\hat{R},R') =$

$\frac{1}{2}|(\hat{R} \triangle R') \cup (\bar{v}\hat{R} \triangle \bar{v}R')| < \frac{1}{2}|(R \triangle \hat{R}) \cup (\bar{v}R \triangle \bar{v}\hat{R})| = k$.

Hence, by the induction hypothesis it follows that:

$\tilde{c}(\hat{R},R') = \tilde{l}(\hat{R},R')$ and $\tilde{c}(R',R) = \tilde{l}(R',R) = 1$.

Using the triangle inequality it follows:

$\tilde{l}(R,\hat{R}) \leq \tilde{l}(R',R) + \tilde{l}(\hat{R},R')$

$\quad = \frac{1}{2}|(R' \triangle R) \cup (\bar{v}R' \triangle \bar{v}R)| + \frac{1}{2}|(R' \triangle \hat{R}) \cup (\bar{v}R' \triangle \bar{v}\hat{R})|$.

Note that $(R' \triangle R) \cup (\bar{v}R' \triangle \bar{v}R) = \{\langle a_i,a_{i+1}\rangle, \langle a_{i+1},a_i\rangle\}$ and $\{\langle a_i,a_{i+1}\rangle, \langle a_{i+1},a_i\rangle\} \cap [(R' \triangle \hat{R}) \cup (\bar{v}R' \triangle \bar{v}\hat{R})] = \varphi$.

Hence, $\tilde{l}(R,\hat{R}) \leq \frac{1}{2}|(R \triangle \hat{R}) \cup (\bar{v}R \triangle \bar{v}\hat{R})|$.

■

In general $\tilde{c}_n|_{Q_n(A) \times Q_n(A)}(r,\hat{r}) \leq \tilde{q}_n(r,\hat{r})$ for all $r,\hat{r} \in Q_n(A)$. By 4.4.13 if follows that if $n = 1$ and $R,\hat{R} \in L(A)$ $\tilde{c}(R,\hat{R}) = \tilde{q}(R,\hat{R}) = \tilde{l}(R,\hat{R})$. Hence, $\tilde{q}$ and $\tilde{c}$ are extensions of $\tilde{l}$. The question is are they different on $Q(A) \times Q(A)$ the domain of $\tilde{q}$. This will be answered in the following example.

Example 4.4.14

In general for all $r,\hat{r} \in Q_n(A)$ $\tilde{c}_n(r,\hat{r}) \leq \tilde{q}_n(r,\hat{r})$.

Moreover $\tilde{c}((xz)y,y(xz)) = 2$ and

$\tilde{q}((xz)y,y(xz)) = 3$ (See example 4.4.7).

Hence, in general for all n, $\tilde{c}_n \neq \tilde{q}_n$ on $Q_n(A)$.

■

It is still an open question whether or not $\tilde{q}_n$ differs from $\tilde{c}_n$ on $L_n(A)$ and whether or not $\tilde{q}_n$ differs from $\tilde{l}_n$ on $L_n(A)$.

For each introduced distance function we are now able to define non-expansiveness conditions according to definition 3.4.4. We will now deduce some relations between these non-expansiveness conditions. Furthermore, it is shown that the independence of irrelevant alternatives implies those non-expansiveness conditions.

<u>Theorem 4.4.15</u>

Suppose V and W are classified sets of orderings, $N = \{1,2,..,n\}$, $A \in Æ$, and F is a welfare function on $\Gamma = \langle A,N \rangle$ from $V_n(A)$ to $W(A)$.

4.4.15.1 If $V_n(A) = L_n(A)$, $W(A) \subseteq C(A)$ and F is independent of irrelevant alternatives, then F is $\langle l_n,c \rangle$-non-expansive.

4.4.15.2 If $V_n(A) \subseteq Q(A)$, $\langle V_n(A),q_n \rangle$ is a full metric space, $W(A) \subseteq C(A)$ and F is independent of irrelevant alternatives, then F is $\langle q_n,c \rangle$-non-expansive.

4.4.15.3 If $V_n(A) \subseteq C_n(A)$, $\langle V_n(A),c_n \rangle$ is a full metric space, $W(A) \subseteq C(A)$ and F is independent of irrelevant alternatives, then F is $\langle c_n,c \rangle$-non-expansive.

4.4.15.4 If $\langle V_n(A),d_n \rangle$ is a metric space, $W(A) \subseteq Q(A)$ and F is $\langle d_n,q \rangle$-non-expansive, then F is $\langle d_n,c \rangle$-non-expansive.

4.4.15.5 If $\langle V_n(A),d_n \rangle$ is a metric space, $W(A) \subseteq L(A)$ and F is $\langle d_n,l \rangle$-non-expansive, then F is $\langle d_n,q \rangle$-non-expansive.

4.4.15.6 If $V_n(A) \subseteq Q_n(A)$, $\langle W(A),d \rangle$ is a metric space and F is $\langle c_n,d \rangle$-non-expansive, then F is $\langle q_n,d \rangle$-non-expansive.

4.4.15.7 If $V_n(A) = L_n(A)$, $\langle W(A),d \rangle$ is a metric space and F is $\langle q_n,d \rangle$-non-expansive, then F is $\langle l_n,d \rangle$-non-expansive.

<u>Proof of theorem 4.4.15</u>

(4.4.15.1),(4.4.15.2) and (4.4.15.3) It suffices by theorem 3.4.8 to prove that, if $\langle r,r' \rangle \in EC(V_n(A),\{x,y\})$, then $\langle F(r),F(r') \rangle \in EC(W(A),\{xy\})$.

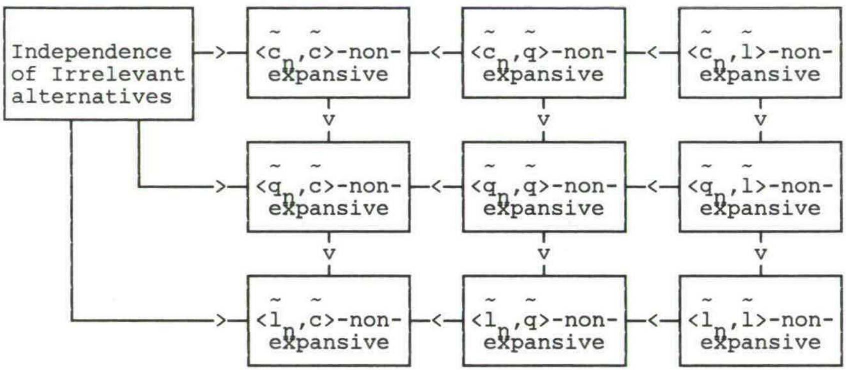
Suppose $\langle r,r' \rangle \in EC(V_n(A),\{x,y\})$, $Ax = A - \{x\}$ and $Ay = A - \{y\}$.

Since $\langle r,r' \rangle \in EC(V_n(A),\{x,y\})$ is equivalent to $r|_{Ax} = r'|_{Ax}$ and $r|_{Ay} = r'|_{Ay}$. It follows from the independence of irrelevant alternatives condition of F, that $F(r)|_{Ax} = F(r')|_{Ax}$ and $F(r)|_{Ay} = F(r')|_{Ay}$. This is again equivalent to $\langle F(r),F(r')\rangle \in EC(W(A),\{xy\})$.

All the other theorems follow immediate from (4.4.13.1) and (4.4.13.2).

■

If we take implicitly the appropriate domains and codomains we can deduce the following implication structure from theorem 4.4.15.



The proof of theorem 4.4.15 clarifies the difference between the non-expansiveness conditions and the independence condition. The latter condition implies that for all $x,y \in A$ elementary changes in $\{x,y\}$ are mapped on elementary changes in $\{x,y\}$, a concequence which is not only necessary but also sufficient (See e.g. Blau [1971]). In the former condition however it is possible that an elementary change in $\{x,y\}$ is mapped on one in $\{a,b\}$.

By theorem 4.4.15 it follows that the independence condition can be replaced by a non-expansiveness conditions in order to weaken the assumptions about welfare functions. Since this independence of irrelevant alternatives and strongly Pareto-optimality do not exclude each other (e.g. take a suitable dictatorship), it follows by theorem 4.4.15 that the non-expansiveness conditions do not have the draw-backs exposed in example 4.4.1. This is a reason to investigate whether or not impossibilities, as deduced in §4.3, can be avoided by substituting non-expansiveness for the independence condition. We already pointed out that this substitution does not help in avoiding dictatorship. The proofs of these new impossibility theorems make use of the notions introduced in §3.5. Before deducing any impossibility result, some preparations involving standard neighbourhoods are needed. We reconsider neighbourhood graphs.

Recall example 4.4.7 and 4.4.10. We start with a notational convention. Let $x,y \in A$. Then $Q(A,x>y) := \{R \in Q(A) : x > y : R\}$. $Q(A,x>y)$ is the set of quasi-ordering in which x is strictly prefered to y. In the same way $V(A,x>y)$ can be defined for any classified set V of orderings. If $A = \{x,y,z\}$, then

$$Q(A,x>y) = \{xyz,\ xzy,\ zxy,\ x(yz), \begin{bmatrix} xy \\ (xz) \\ (yz) \end{bmatrix}, (xz)y\}.$$

It is in the left most part of the graph in (4.4.7). $L(A,x>y)$ is in the right half of the graph in (4.4.10).

From the graphs of example 4.4.10 and 4.4.7 we learn that a relation R is in the neighbourhood of $\langle V(A,x>y),V(A,y>x)\rangle$, iff there are $R_{xy} \in V(A,x>y)$ and $R_{yx} \in V(A,y>x)$, such that $\langle R,R_{xy}\rangle$, $\langle R,R_{yx}\rangle$ and $\langle R_{xy}, R_{yx}\rangle \in EC(V(A), \{x,y\})$. Hence, R, R_{xy} and R_{yx} are elementary changes of each other.

Theorem 4.4.16

Suppose V is a classified set of orderings, $A \in \mathcal{Æ}$, x and y in A and $M = \langle V(A),d\rangle$ a metric space, with meshwidth mesh, such that for all $R, R' \in V(A)$:

$d(R,R') = \text{mesh}$, iff $\langle R,R'\rangle \in \bar{n}EC(V(A))$.

Then the neighbourhood $NH_M(V(A,x>y),V(A,y>x)) =$

$\{R \in V(A)$: there is a $R_{xy} \in V(A,x>y)$ and a $R_{yx} \in V(A,y>x)$, such that $\{\langle R,R_{xy}\rangle,\langle R,R_{yx}\rangle,\langle R_{xy},R_{yx}\rangle\} \subseteq EC(V(A),\{x,y\})\ \}$.

Proof of theorem 4.4.16

Let V,x,y,A and M be as above.

We only prove the inclusion "$\subseteq$", the other inclusion is trivial by the definitions and properties of d.

If $R \in NH_M(V(A,x>y),V(A,y>x))$, then there are $R^e_{xy} \in V(A,x>y)$ and $R^e_{yx} \in V(A,y>x)$, such that $d(R,R^e_{xy}) < e$ and $d(R,R^e_{yx}) < e$ for all $e > \text{mesh}$. Since $V(A)$ is finite there must be $R_{xy} \in V(A,x>y)$ and $R_{yx} \in V(A,y>x)$, such that $d(R,R_{xy}) \le \text{mesh}$ and $d(R,R_{yx}) \le \text{mesh}$.

Since $R_{xy} \neq R_{yx}$ let without loss of generality $R \neq R_{xy}$.

Then obviously $<R,R_{xy}> \varepsilon EC(V(A))$.

Now we have two cases:

Case 1 $R = R_{yx}$. Then $<R_{xy},R_{yx}> \varepsilon \bar{n}EC(V(A),\{x,y\})$ and we are done.

Case 2 $R \neq R_{yx}$. Then $<R_{yx},R> \varepsilon EC(V(A),\{a,b\})$ and $<R,R_{xy}> \varepsilon EC(V(A),\{c,d\})$, for some $a,b,c,d \varepsilon A$.

Subcase 2A $\{a,b\} \neq \{c,d\}$. Then $\{x,y\} \neq \{a,b\}$ or $\{x,y\} \neq \{c,d\}$.

Without loss of generality suppose $\{x,y\} \neq \{a,b\}$.

Then $y > x : R \varepsilon V(A,y>x)$, since $E_{ab} \cap R = E_{ab} \cap R_{yx}$.

So $\{c,d\} = \{x,y\}$ and we are done by taking R_{xy} and $R = R_{yx}$.

Subcase 2B $\{a,b\} = \{c,d\}$. Since $EC(V(a,b),\{a,b\})$ is an equivalence relation, we are done.

■

In the next theorem we prove that these neighbourhoods of $<V(A,x>y),V(A,y>x)>$ almost determine a linear ordering. For every $R \varepsilon L(A)$, such that $x_1x_2...x_p : R$, there are $R_{x_{i+1}x_i} \varepsilon Q(A)$, such $<R,R_{x_{i+1}x_i}> \varepsilon \bar{n}EC(Q(A),\{x_i,x_{i+1}\})$, for all $i \varepsilon \{1,2,...p-1\}$. Hence, for each $R \varepsilon L(A)$ there are (p-1) different types of non-trivial elementary change in which R is involved. Consider the sets of elementary changes in which an ordering $R' \varepsilon Q(A)$ is non-trivially involved. From the pictures of example 4.4.7 and 4.4.10 one can see, if we have (p-1) different elementary changes, then there are precisely three relations $R^1,R^2,R^3 \varepsilon Q(A)$, which are non-trivially involved in all those changes. Furthermore, $\{R^1,R^2,R^3\}$ has the form $\{R,\bar{v}R,A \times A\}$, with $R \varepsilon L(A)$. This pictorial obtained result is formally verified in the next theorem. The previous theorem together with theorem 4.4.18 and corollary 3.5.11 points out our goal.

Theorem 4.4.17

Suppose V is a classified set of ordeings, $A \varepsilon Æ$, $R \varepsilon L(A)$, $\hat{R} \varepsilon V(A)$, $A = \{a_1,a_2,...,a_p\}$, $a_1a_2....a_p : R$, $V(A) \subseteq Q(A)$, $M = <V(A),d>$ is a metric space, with meshwidth mesh, such that for all $R',R'' \varepsilon V(A)$:

$d(R'',R') =$ mesh, iff $<R',R''> \varepsilon \bar{n}EC(V(A))$.

If $\hat{R} \varepsilon NH_M(V(A,a_i>a_{i+1}),V(A,a_{i+1}>a_i))$ for all $i \varepsilon \{1,..p-1\}$, then $\hat{R} \varepsilon \{R,\bar{v}R,A \times A\}$.

Proof of theorem 4.4.17

Suppose all variables are as above and for all $i \in \{1,..p-1\}$ $\hat{R} \in NH_M(V(A,a_i>a_{i+1}),V(A,a_{i+1}>a_i))$.

Hence, for all $i \in \{1,\hat{2},3,...,p-1\}$ it follows by theorem 4.4.16 that there are $\hat{R}_{a_ia_{i+1}} \in V(A,a_i>a_{i+1})$ and $\hat{R}_{a_{i+1}a_i} \in V(A,a_{i+1}>a_i)$, such that $\langle\hat{R},\hat{R}_{a_ia_{i+1}}\rangle$, $\langle\hat{R}_{a_{i+1}a_i},\hat{R}\rangle$, $\langle\hat{R}_{a_{i+1}a_i},\hat{R}_{a_ia_{i+1}}\rangle \in EC(V(A),\{a_i,a_{i+1}\})$. Moreover, by (4.4.4) it then follows for all $i \in \{1,2,...,p-1\}$ that

$a_i \sim a_{i+1} : suc\hat{R}$, $a_i \sim a_{i+1} : suc\hat{R}a_ia_{i+1}$, $a_i \sim a_{i+1} : suc\hat{R}a_{i+1}a_i$.

Now we prove two claims.

Claim 4.4.17.1 For all $i \in \{1,...p-1\}$ and all $x \in A - \{a_i,a_{i+1}\}$:

4.4.17.1a $a_i > x : \hat{R}$, iff $a_{i+1} > x : \hat{R}$, and

4.4.17.1b $x < a_i : \hat{R}$, iff $x < a_{i+1} : \hat{R}$.

Proof of claim 4.4.17.1

(Only if of 4.4.17.1a) Suppose $a_i > x : \hat{R}$.

Since $x \notin \{a_{i+1},a_i\}$ it follows from $\langle\hat{R}, \hat{R}_{a_{i+1}a_i}\rangle \in EC(V(A),\{a_i,a_{i+1}\})$ that $a_i > x : \hat{R}_{a_{i+1}a_i}$.

Hence, by the $\langle\bar{a}^2,\bar{a}\rangle$-transitivity of $\hat{R}_{a_{i+1}a_i}$ it follows:

$a_{i+1} > x : \hat{R}_{a_{i+1}a_i}$, which yields $a_{i+1} > x : \hat{R}$, with $\langle\hat{R},\hat{R}_{a_{i+1}a_i}\rangle \in EC(V(A),\{a_i,a_{i+1}\})$.

All the other implications are similar.

Claim 4.4.17.2 For all $i \in \{1,...p-2\}$:

4.4.17.2a $a_i > a_{i+1} : \hat{R}$, iff $a_{i+1} > a_{i+2} : \hat{R}$, and

4.4.17.2b $a_{i+1} > a_i : \hat{R}$, iff $a_{i+2} > a_{i+1} : \hat{R}$.

Proof of claim 4.4.17.2

(Only if of 4.4.17.2a) Suppose $a_i > a_{i+1} : \hat{R}$ and $a_{i+2} \geq a_{i+1} : \hat{R}$.

By (4.4.17.1) it follows $a_i > a_{i+2} : \hat{R}$ and $a_{i+2} \geq a_i : \hat{R}$.

This cannot be the case.

Similarly the other implications can be deduced.

We will now finish the proof of theorem 4.4.17.

We distinguish two cases.

Case 1 There is an index i with $a_i > a_{i+1} : \hat{R}$ or $a_{i+1} > a_i : \hat{R}$.

By a successive use of claim 4.4.8.2 it follows:

$\hat{R} \in \{R, \bar{v}R\}$ and we are done.

<u>Case 2</u> For all $i \in \{1,2,...,p\}$ $a_i \sim a_{i+1} : \hat{R}$.

It is sufficient to prove that $\hat{R} = A \times A$.

Suppose $a_i > a_j : \hat{R}$.

Without loss of generality suppose $i < j$.

Then $j > i + 1$ and by (4.4.17.1a) it follows $a_{i+1} < a_j$.

Hence, $j > i + 2$. Repetition of this reason leads obviously to a contradiction, since $|A| = p < \infty$.

■

We will now prove some other prelimenary lemma's in order to apply corollary 3.5.11.

<u>Lemma 4.4.18</u>

Suppose $A \in Æ$, with $|A| = p$, and $x,y \in A$.

4.4.18.1 If $R^1 \in L(A)$ and $R^2 \in Q(A)$, then $\tilde{c}(R^1,R^2) = |R^2 \cap \bar{c}R^1|$.

4.4.18.2 If $R^1 \in L(A)$ and $R^2,R^3 \in Q(A)$, such that $R^2 \subseteq R^3$, then $\tilde{c}(R^1,R^3) \geq \tilde{c}(R^1,R^2)$.

4.4.18.3 If $R^1 \in L(A)$ and $R^2 \in Q(A)$, such that $\bar{a}R^2 \subseteq \bar{v}R^1$, then $\tilde{c}(R^1,R^2) = \binom{p}{2}$.

4.4.18.4 If $R^1,R^2,R^3 \in L(A,x>y)$, with $\langle R^3,\bar{v}R^1\rangle \in EC(L(A),\{xy\})$, then $\tilde{c}(R^1,R^2) + \tilde{c}(R^2,R^3) = \tilde{c}(R^1,R^3) = \binom{p}{2} -1$.

4.4.18.5 For all $R,\hat{R} \in L(A)$ and for all $\tilde{R} \in Q(A)$: $\tilde{R} \in [R,\hat{R}]_{\langle Q(A),\tilde{c}\rangle}$, iff $\tilde{R} \in L(A)$ & $R \cap \hat{R} \subseteq \tilde{R}$.

<u>Proof of lemma 4.4.18</u>

Let $A \in Æ$ be such that $|A| = p$ and let $x,y \in A$.

(4.4.18.1.) Suppose $R^1 \in L(A)$ and $R^2 \in Q(A)$. Then

$$\begin{aligned}
\tilde{c}(R^1,R^2) &= \tfrac{1}{2}|(R^1-R^2) \cup (R^2-R^1) \cup (\bar{v}R^1-\bar{v}R^2) \cup (\bar{v}R^2-\bar{v}R^1)| \\
&= \tfrac{1}{2}|(R^1\cap\bar{c}R^2) \cup (R^2\cap\bar{c}R^1) \cup (\bar{v}R^1\cap\overline{cv}R^2) \cup (\bar{v}R^2\cap\overline{cv}R^1)| \\
&= \tfrac{1}{2}|(\bar{n}R^1\cap\bar{c}R^2) \cup (R^2\cap\bar{c}R^1) \cup (\bar{c}R^1\cap\overline{cv}R^2) \cup (\bar{v}R^2\cap\bar{n}R^1)| \\
&= \tfrac{1}{2}|\{\bar{n}R^1 \cap (\bar{c}R^2 \cup \bar{v}R^2)\} \cup \{\bar{c}R^1 \cap (R^2 \cup \overline{cv}R^2)\}| \\
&= \tfrac{1}{2}|\{\bar{n}R^1 \cap \bar{v}R^2\} \cup \{\overline{vn}R^1 \cap R^2\}| \\
&= |\overline{vn}R^1 \cap R^2| = |R^2 \cap \bar{c}R^1|.
\end{aligned}$$

(4.4.18.2) Suppose $R^1 \in L(A)$ and $R^2,R^3 \in Q(A)$, such that $R^2 \subseteq R^3$.

Then $\tilde{c}(R^1,R^3) = |R^3 \cap \bar{c}R^1| \geq |R^2 \cap \bar{c}R^1| = \tilde{c}(R^1,R^2)$.

(4.4.18.3) Suppose $R^1 \in L(A)$ and $R^2 \in Q(A)$, such that $\bar{a}R^2 \subseteq \bar{v}R^1$.

Then $\bar{v}R^1 \subseteq R^2$.

So $\tilde{c}(R^1,R^2) \geq \tilde{c}(R^1,\bar{v}R^1) = |\bar{v}R^1 \cap \bar{c}R^1| = |\bar{n}R^1| = \binom{p}{2}$.

But $\tilde{c}(R^1,R^2) \leq |\{\{x,y\} : x \neq y,\ x,y \in A\}| = \binom{p}{2}$.
Hence, $\tilde{c}(R^1,R^2) = \binom{p}{2}$.

(4.4.18.4) Suppose $R^1,R^2,R^3 \in L(A,x>y)$, such that $\langle R^3,\bar{v}R^1\rangle \in EC(L(A),\{x,y\})$.

Then $\tilde{c}(R^1,R^3) = |R^3 \cap \bar{c}R^1| = |R^3 \cap \overline{vn}R^1| \geq \binom{p}{2} - 1$.

But by the triangle inequality we have:

$$\begin{aligned}
\tilde{c}(R^1,R^3) &\leq \tilde{c}(R^1,R^2) + \tilde{c}(R^2,R^3)\\
&= |R^2 \cap \bar{c}R^1| + |R^2 \cap \bar{c}R^3|\\
&= |R^2 \cap \bar{c}R^1| + |R^2 \cap \bar{c}((\bar{v}R^1 - \{\langle y,x\rangle\}) \cup \{\langle x,y\rangle\})|\\
&= |R^2 \cap \bar{c}R^1| + |R^2 \cap (((\bar{n}R^1) - \{\langle x,y\rangle\}) \cup \{\langle y,x\rangle\})|\\
&= |R^2 \cap \bar{c}R^1| + |R^2 \cap (\bar{n}R^1 - \{\langle x,y\rangle\})|\\
&= |R^2 \cap (\bar{n}(A \times A) - \{\langle x,y\rangle\})| = \binom{p}{2} - 1.
\end{aligned}$$

(4.4.18.5) Let $R,\hat{R} \in L(A)$ and $\tilde{R} \in Q(A)$.

$\tilde{R} \in [R,\hat{R}]_{\langle Q(A),\tilde{c}\rangle}$, iff

$\tilde{c}(R,\tilde{R}) + \tilde{c}(\tilde{R},\hat{R}) = \tilde{c}(R,\hat{R})$, iff

$\binom{p}{2} - |\bar{n}(R \cap \hat{R})| \leq |\tilde{R} \cap \bar{c}(R \cap \hat{R})| = |\tilde{R} \cap (\bar{c}R \cup \bar{c}\hat{R})|$
$\leq |\tilde{R} \cap \bar{c}R| + |\tilde{R} \cap \bar{c}\hat{R})| = |R \cap \bar{c}\hat{R})| = \binom{p}{2} - |\bar{n}(R \cap \hat{R})|$, iff
$\tilde{R}$ is antisymmetric (because of the first inequality being an equality), $\tilde{R} \in Q(A)$ and $R \cap \hat{R} \subseteq \tilde{R}$ (because of the second inequality being an equality).

■

Enough preparations have been made for the next impossibility theorem. To avoid recapitulation of assumptions in the sequence of theorems leading to the impossibility some assumptions are stated here.

<u>Assumption 4.4.19</u>

Suppose $\Gamma = \langle A,N\rangle$ is a society, with $|A| = p$ and $N = \{1,2,\ldots,n\}$, V is a classified set of orderings, $V(A) \subseteq Q(A)$, and F is a $\langle \tilde{1}_n,\tilde{c}\rangle$-non-expansive and Pareto-optimal welfare function on Γ from $L_n(A)$ to $V(A)$.

■

As a notational convention, we denote by $\langle R^S,\hat{R}^{N-S}\rangle$ the profile $\langle R^1,R^2,\ldots,R^n\rangle \in W_n(A)$, such that for all $i \in S$ $R^i = R$ and for all $i \in N - S$ $R^i = \hat{R}$, where R and $\hat{R}$ are in $W(A)$ a classified set of orderings restricted to A.

So if $S = \{1,2,...,s\} \subseteq N$, then $\langle \underbrace{R,R,....,R}_{s\text{-times}},\underbrace{\hat{R},\hat{R},...,\hat{R}}_{(n-s)\text{-times}} \rangle$.

Let $S \subseteq N$ and $R \in L(A)$ it is obvious that $\langle R^S, \bar{v}R^{N-S} \rangle$ is a profile in which the two coalitions S and N-S oppose each other totally. Such profiles will be called <u>maximal conflicts between S and</u> N-S. Now every person who has to arbitrate in such a maximal conflict situation, will most probably come up with a compromizing ordering $\hat{R}$ between R and $\bar{v}R$, where the resemblance of $\hat{R}$ and R would depend on the ratio $|S| \div |N|$. The assumed genelalized welfare function F in (4.4.19) is not consistent with this procedure. See the following lemma. Hence, one may already perceive the occurence of an impossibility.

<u>Lemma 4.4.20</u>

Assume (4.4.19), $S \subseteq N$ and $R \in L(A)$.
Then $F(\langle R^S, \bar{v}R^{N-S} \rangle) \in \{R, \bar{v}R, A \times A\}$.

<u>Proof of lemma 4.4.20</u>

Let $a_1 a_2 a_3 ... a_p : R$. By theorem 4.4.17 it is sufficient to prove that $F(\langle R^S, \bar{v}R^{N-S} \rangle) \in NH_M(V(A,x>y),V(\underset{\sim}{A},y>x))$, where $\{x,y\} = \{a_i, a_{i+1}\}$ is arbitrary and $M = \langle V(A), c \rangle$.
This is proved by virtue of (3.5.11).
Define $\tilde{F}(\langle \underset{\sim}{R}, \hat{R} \rangle) := F(\tilde{R}^S, \hat{R}^{N-S} \rangle)$ for all $\tilde{R}$, $\hat{R} \in L(A)$.
Obviously $\tilde{F}$ satisfies assumption 3.5.6.
By (4.4.18.4), it follows that L(A,x>y) can be approximated by interiors of ellipses from R and $\bar{v}R$ and radius $(\frac{p}{2})$.
The same holds for L(A,y>x).
Now for all $\tilde{R} \in L(A,x>y)$ and for all $\hat{R} \in V(A,x>y)$, since $\langle x,y \rangle \in \bar{a}\hat{R} \cap \tilde{R}$, we have $\tilde{c}(\tilde{R},\hat{R}) = |\tilde{R} - \bar{c}\hat{R}| \leq |\bar{c}\hat{R}| - 1 < (\frac{p}{2})$.
Furthermore, if $\hat{R} \in V(A) - V(A,x>y)$, there is a linear ordering R", with $R" \in L(A,y>x)$ and $R" \subseteq \hat{R}$. So $\tilde{c}(\bar{v}R",R") = \tilde{c}(\bar{v}R",\hat{R}) = (\frac{p}{2})$ by (4.4.18.2) and $\bar{v}R" \in L(A,x>y)$.
Hence, V(A,x>y) circularly encloses L(A,x>y) with diameter $(\frac{p}{2})$. Similar V(a,x>y) circularly encloses L(A,y>x) with diameter $(\frac{p}{2})$.
This makes (3.5.11) applicable and yields the wanted result.

■

So at a maximal conflict situation it is either that S is decisive, or N-S is decisive, or there is no decision at all.

The rest of the proof towards the impossibility is just a stepwise strenghtening of these decision powers.

Lemma 4.4.21

Assume (4.4.19), with $|A| \geq 3$, and $S \subseteq N$. Then either

for all $R \in L(A)$: $F(\langle R^S, \bar{v}R^{N-S}\rangle) = R$, or

for all $R \in L(A)$: $F(\langle R^S, \bar{v}R^{N-S}\rangle) = \bar{v}R$, or

for all $R \in L(A)$: $F(\langle R^S, \bar{v}R^{N-S}\rangle) = A \times A$.

Proof of lemma 4.4.21

Let $\langle R,\hat{R}\rangle \in EC(L(A),\{x,y\})$.

Suppose $F(\langle R^S, \bar{v}R^{N-S}\rangle) = R$.

By (4.4.20) it is sufficient to prove that $\bar{v}\hat{R} \not\subseteq F(\langle \hat{R}^S, \bar{v}\hat{R}^{N-S}\rangle)$.

Suppose $\bar{v}\hat{R} \subseteq F(\langle \hat{R}^S, \bar{v}\hat{R}^{N-S}\rangle)$.

Note $\tilde{l}_n(\langle R^S, \bar{v}R^{N-S}\rangle, \langle \hat{R}^S, \bar{v}\hat{R}^{N-S}\rangle) = 1$.

But $\tilde{c}(F(\langle R^S, \bar{v}R^{N-S}\rangle), F(\langle \hat{R}^S, \bar{v}\hat{R}^{N-S}\rangle)) \geq \tilde{c}(R, \bar{v}\hat{R}) = \binom{p}{2} - 1 \geq 5.$

(4.4.21.1)

Hence, F is not $\langle \tilde{l}_n, \tilde{c}\rangle$-non-expansive, which contradicts our assumption 4.4.19. We are done.

■

Lemma 4.4.21 does not hold for p = 2 since then the inequality at (4.4.21.1) obviously is not true. Furthermore, this lemma shows that at all maximal conflict situations either S is decisive, ór N-S is decisive, ór there is no decision at all, which actually strengthens lemma 4.4.20. One might reason that whenever S decides on all maximal conflict situations, where N-S opposes most firmly, S is decisive at any profile, where the opposition is not so strong. This reasoning is based on monotonicity property, which although being natural, is not imposed on F. Next this monotonicity is proved.

Lemma 4.4.22

Assume (4.4.19) and $S \subseteq N$.

Furthermore, suppose for all $R \in L(A)$: $F(\langle R^S, \bar{v}R^{N-S}\rangle) = R$.

Then for all $R, \hat{R} \in L(A)$: $F(\langle R^S, \bar{v}\hat{R}^{N-S}\rangle) = R$.

Proof of lemma 4.4.22

By induction on $k \geq 0$ it suffices to prove for all $R,\hat{R} \in L(A)$: if $\tilde{c}(R,\hat{R}) \leq k$, then $F(\langle R^S,\hat{R}^{N-S}\rangle) = R$.

Basis: Is trivial by the Pareto-optimality.

Induction step: Let $R,\hat{R} \in L(A)$, with $\tilde{c}(R,\hat{R}) = k+1$.

Since $\langle L(A),\tilde{c}\rangle$ is a full metric space there is a path of elementary changes from $\bar{v}\hat{R}$ to $\hat{R}$ via R.

Without loss of generality suppose the following picture:

$$\bar{v}\hat{R} \underset{\binom{p}{2}-k-1}{\text{———}} R \underset{1}{\text{———}} \bar{R} \underset{k-1}{\text{———}} \bar{\bar{R}} \underset{1}{\text{———}} \hat{R}$$

(If $k=0$, then $\bar{R} = \bar{\bar{R}} = \hat{R}$. If $k=1$, then $\bar{R} = \bar{\bar{R}}$)

Obviously it holds that:

$$\begin{array}{ccccccc}
\cdot & \overset{\binom{p}{2}-k-1}{\text{————}} & \cdot & \overset{1}{\text{————}} & \cdot & \overset{k}{\text{————}} & \cdot \\
\langle \bar{v}\hat{R}^S,\hat{R}^{N-S}\rangle & & 1 \Big| & \langle R^S,\hat{R}^{N-S}\rangle & & \langle \bar{R}^S,\hat{R}^{N-S}\rangle & \langle \hat{R}^S,\hat{R}^{N-S}\rangle \\
 & & \cdot & \langle R^S,\bar{\bar{R}}^{N-S}\rangle & & &
\end{array}$$

By the induction hypothesis $F(\langle \bar{R}^S,\bar{R}^{N-S}\rangle) = \bar{R}$ and $F(\langle R^S,\bar{\bar{R}}^{N-S}\rangle) = R$.

By the assumptions $F(\langle \bar{v}\hat{R}^S,\hat{R}^{N-S}\rangle) = \bar{v}\hat{R}$.

Using the non-expansiveness of F we have:

$\tilde{R} := F(\langle R^S,\hat{R}^{N-S}\rangle) \in [\bar{v}\hat{R},\bar{R}]_{\langle Q(A),\tilde{c}\rangle}$.

Hence, by (4.4.18.4) $\tilde{R} \in L(A)$ and $\bar{v}\hat{R} \cap \bar{R} \subseteq \tilde{R}$.

However since $\langle \tilde{R},R\rangle$, $\langle R,\bar{R}\rangle$, $\langle \bar{R},R\rangle \in EC(L(A))$ it follows by (4.4.4.3) that $\tilde{R} = R$, since $\bar{R} \neq R$.

■

Lemma 4.4.23

Assume (4.4.19) and $S \subseteq N$.

Furthermore, suppose for $R \in L(A)$: $F(\langle R^S,\bar{v}R^{N-S}\rangle) = A \times A$.

Then for all $R,\hat{R} \in L(A)$: $F(\langle R^S,\hat{R}^{N-S}\rangle) = R \cup \hat{R}$.

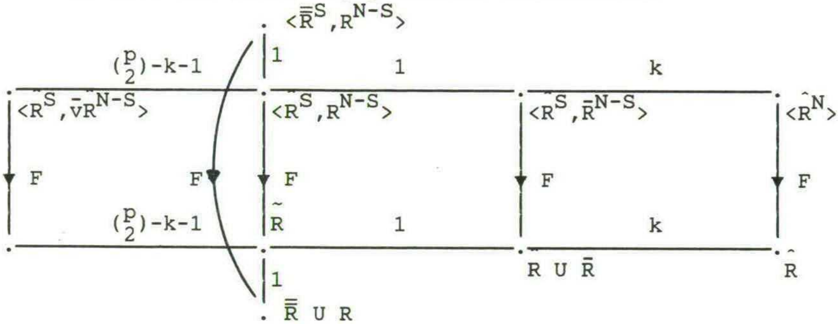
Proof of lemma 4.4.23

It is sufficient to prove by induction on $k \geq 0$ that for all $\tilde{R},\hat{R} \in L(A)$, with $c(\tilde{R},\hat{R}) = k$: $F(\langle \hat{R}^S,\tilde{R}^{N-S}\rangle) = \tilde{R} \cup \hat{R}$.

Basis: is trivial by the Pareto-optimality.

Induction step: let $\tilde{R},\hat{R} \in L(A)$, such that $c(\tilde{R},\hat{R}) = k + 1$.

We have the following pictorial information:



Where $\bar{R} = (\tilde{R} - \{\langle a,b\rangle\}) \cup \{\langle b,a\rangle\}$, $a > b : \tilde{R}$ and $b > a : \hat{R}$

$\bar{\bar{R}} = (\hat{R} - \{\langle x,y\rangle\}) \cup \{\langle y,x\rangle\}$, $x > y : \hat{R}$ and $y > x : \tilde{R}$.

There are two cases:

Case 1 $\{a,b\} \neq \{x,y\}$.

Obviously $b > a : \bar{R} \cup \hat{R}$ and $x \sim y : \bar{R} \cup \hat{R}$ and $a \sim b : \bar{\bar{R}} \cup \tilde{R}$ and $y > x : \bar{\bar{R}} \cup \tilde{R}$.

Since $c(\tilde{R},\bar{R}) = c(\tilde{R},\bar{\bar{R}}) = 1$ it follows that $a \sim b : \tilde{R}$ and $x \sim y : \tilde{R}$.

Hence, $F(\langle \hat{R}^S,\tilde{R}^{N-S}\rangle) = \tilde{R} \cup \hat{R}$.

Case 2 $\{a,b\} = \{x,y\}$.

Obviously $b > a : \bar{R} \cup \hat{R}$, $a > b : \bar{\bar{R}} \cup \tilde{R}$, $\langle \bar{R} \cup \hat{R}, \bar{\bar{R}} \cup \tilde{R}\rangle \in EC(Q(A))$, which again leads to $F(\langle \hat{R}^S,\tilde{R}^{N-S}\rangle) = \tilde{R} \cup \hat{R}$.

■

Lemma 4.4.24

Assume (4.4.19), $\{n,n-1,n-2,\ldots,n-s+1\} = S \subseteq N$, and for all $R,\hat{R} \in L(A)$: $F(\langle R^S,\hat{R}^{N-S}\rangle) = R$.

Then for all $r \in L_n(A)$ and $R \in L(A)$:

if for all $i \in S$: $R^i = R$, then $F(r) = R$.

Proof of lemma 4.4.25

Let $r \in L_{n-s}(A)$ and $R \in L(A)$. By the symbols $\langle r,R^S\rangle$ we denote the profile $\tilde{r} \in L_n(A)$ such that $\tilde{R}^i = R^i$ for all $i \in N - S$ and $\tilde{R}^i = R$ for all $i \in S$.

Suppose $\langle r,\bar{\bar{r}}\rangle \in EC(L_{n-s}(A),\{x,y\})$ and for all $R \in L(A)$

$F(\langle r,R^S\rangle) = R$.

It is sufficient to prove that $F(\langle\bar{\bar{r}},R^S\rangle) = R$ for all $R \in L(A)$.

Take $R_1,R_2 \in L(A)$ such that $\langle R_1,R_2\rangle \in EC(L(A),\{x,y\})$ $x > y : R_1$ and $y > x : R_2$.

Take a shortest path $R_1,R_2,R_3,\ldots R_{\binom{p}{2}} = \bar{v}R_1$ in $L(A)$ of elementary changes.

Denote $F(\langle\bar{\bar{r}},R_i{}^S\rangle) = \bar{\bar{R}}_i$.

It suffices to prove $\bar{\bar{R}}_i = R_i$, for all $i \in \{1,2,\ldots,\binom{p}{2}\}$.

We have now the following picture of elementary changes:

$$\begin{tikzcd}
\bar{\bar{R}}_1 \arrow[r, dash] \arrow[d, dash] \arrow[dr, dash] & \bar{\bar{R}}_2 \arrow[r, dash] \arrow[d, dash] \arrow[dl, dash] & \bar{\bar{R}}_3 \arrow[r, dash] \arrow[d, dash] & \bar{\bar{R}}_4 \arrow[r, dash] \arrow[d, dash] & \cdots \arrow[r, dash] & \bar{\bar{R}}_{\binom{p}{2}-1} \arrow[r, dash] \arrow[d, dash] & \bar{\bar{R}}_{\binom{p}{2}} \arrow[r, dash] \arrow[d, dash] \arrow[dr, dash] & \bar{v}\bar{\bar{R}}_2 \arrow[d, dash] \arrow[dl, dash] \\
R_1 \arrow[r, dash] & R_2 \arrow[r, dash] & R_3 \arrow[r, dash] & R_4 \arrow[r, dash] & \cdots \arrow[r, dash] & R_{\binom{p}{2}-1} \arrow[r, dash] & R_{\binom{p}{2}} \arrow[r, dash] & \bar{v}R_2
\end{tikzcd}$$

Here an arc means that its end points are elementary changes.

We continue by first proving a claim.

Claim 4.4.24.1 Let $X_0,X_1,X_2,\ldots,X_k$ be a shortest path of elementaire changes in $Q(A)$, such that $X_0,X_k \in L(A)$, $\tilde{c}(X_0,X_k) = k$.

Suppose $\langle X_0,Y\rangle,\langle Y,X_1\rangle \in EC(Q(A))$ and $Y \in L(A)$.

Then $X_1 = Y$.

Proof of claim 4.4.24.1

We have
$$\begin{tikzcd}
X_0 \arrow[r, dash] \arrow[dr, dash] & X_1 \arrow[r, dash] \arrow[d, dash] & X_0 \arrow[r, dash] & \cdots \arrow[r, dash] & X_k \\
& Y & & &
\end{tikzcd}$$

Since $\langle X_0,X_1\rangle$, $\langle X_1,Y\rangle$, $\langle Y,X_0\rangle \in EC(Q(A))$ it follows that $\langle X_0,X_1\rangle$, $\langle X_1,Y\rangle$, $\langle Y,X_0\rangle \in EC(Q(A),\{a,b\})$ for some $a,b \in A$.

Without loss of generality let $a > b : X_0$ and $b > a : Y$.

Now $b \geq a : X_1$, otherwise $\tilde{c}(X_1,X_k) = \tilde{c}(X_0,X_k)$, which would contradict the shortest walk assumption.

If $b > a : X_1$, then $Y = X_1$.

If $b \sim a : X_1$, then it follows that $\tilde{c}(X_k,X_1) \geq \tilde{c}(X_k,X_0)$, since $X_0 \subseteq X_1$, which contradicts our distance assumption.

This completes the proof of claim 4.4.24.1.

Obviously $\langle R_1,R_2\rangle, \langle R_2,\bar{\bar{R}}_2\rangle, \langle \bar{\bar{R}}_2,R_1\rangle \in EC(Q(A),\{x,y\})$.

Hence, $\bar{\bar{R}}_2 \in \{R_1, R_2, R_1 \cup R_2\}$ and similarly $\bar{\bar{R}}_{\binom{p}{2}} \in \{\bar{v}R_1, \bar{v}R_2, \bar{v}R_1 \cup \bar{v}R_2\}$. Now there are several possibilities each corresponding with a case according to the following diagram:

$\bar{\bar{R}}_{\binom{p}{2}} =$ \ $\bar{\bar{R}}_2 =$	R_1	R_2	$R_1 \cup R_2$
$\bar{v}R_1$	I	II	I
$\bar{v}R_2$	I	I	III
$\bar{v}R_1 \cup \bar{v}R_2$	III	I	I

<u>Case I</u> $\tilde{c}(\bar{\bar{R}}_1,\bar{\bar{R}}_{\binom{p}{2}}) = \binom{p}{2}$. But $\tilde{q}(\bar{\bar{R}}_2,\bar{\bar{R}}_{\binom{p}{2}}) \le \binom{p}{2} - 1$, this contradicts (4.4.13.1).

<u>Case II</u> Since $\tilde{c}(\bar{\bar{R}}_2,\bar{\bar{R}}_{\binom{p}{2}}) = \binom{p}{2}-1$, $R_2,R_3,\ldots,R_{\binom{p}{2}}$ is a shortest path. Using claim 4.4.24.1 succesively we are done.

<u>Case III</u> Now $\bar{\bar{R}}_2 \cap E_{xy} = R_2 \cap E_{xy}$, $\bar{\bar{R}}_{\binom{p}{2}} \cap E_{xy} = R_{\binom{p}{2}} \cap E_{xy}$, $x \ge y : \bar{\bar{R}}_2$ and $x \ge y : \bar{\bar{R}}_{\binom{p}{2}}$.

Hence, for $\langle a,b\rangle \in E_{xy}$, $a \ne b$, it is either $\langle a,b\rangle \in \bar{\bar{R}}_2$ ór $\langle a,b\rangle \in R_{\binom{p}{2}}$.

Suppose $y > x : \bar{\bar{R}}_i$, for some $2 < i < \binom{p}{2}$.

Then $\tilde{c}(\bar{\bar{R}}_2,\bar{\bar{R}}_i) =$

$$= \tfrac{1}{2}|\{\langle a,b\rangle \in \bar{n}E_{xy} : (\langle a,b\rangle \in \bar{\bar{R}}_2 \text{ and } \langle a,b\rangle \notin \bar{\bar{R}}_i) \text{ or } (\langle a,b\rangle \notin \bar{\bar{R}}_2 \text{ and } \langle a,b\rangle \in \bar{\bar{R}}_i)\}| + 1$$

$$= |\{\langle a,b\rangle \in \bar{n}E_{xy} : (\langle a,b\rangle \in \bar{\bar{R}}_i \text{ and } \langle a,b\rangle \notin \bar{\bar{R}}_2)\}| + 1.$$

Similar $\tilde{c}(\bar{\bar{R}}_{\binom{p}{2}},\bar{\bar{R}}_i) =$

$$= |\{\langle a,b\rangle \in \bar{n}E_{xy} : (\langle a,b\rangle \in \bar{\bar{R}}_i \text{ and } \langle a,b\rangle \notin \bar{\bar{R}}_{\binom{p}{2}}\}| + 1.$$

Hence, $\tilde{c}(\bar{\bar{R}}_2,\bar{\bar{R}}_i) + \tilde{c}(\bar{\bar{R}}_i,\bar{\bar{R}}_{\binom{p}{2}}) \ge \tfrac{1}{2}|E_{xy}| + 2 = \binom{p}{2} + 1$.

This contradicts the fact that $\bar{\bar{R}}_i$ is on a shortest path from $\bar{\bar{R}}_2$ to $\bar{\bar{R}}_{\binom{p}{2}}$ and therefor we have for all $i \in \{2,3,\ldots,\binom{p}{2}\}$

that $x \geq y : \bar{\bar{R}}_i$.

Note that $y > x : R_i$ for $i \in \{2,3,\dots,\binom{p}{2}\}$.

Since $\langle\bar{\bar{R}}_i, R_i\rangle \in EC(Q(A))$ it follows that $\langle\bar{\bar{R}}_i, R_i\rangle \in EC(Q(A),\{x,y\})$.

Suppose $y > a > x : R_i$ for some $i \in \{2,3,\dots,\binom{p}{2}\}$.

Then, since $R_i \cap E_{xy}$, it follows $a > x : \bar{\bar{R}}_i$ and $y > a : \bar{\bar{R}}_i$.

Hence, $\bar{\bar{R}}_i$ is not $\langle\bar{a}^2,\bar{a}\rangle$-transitive.

So: not $y > a > x : R_i$ for all $a \in A - \{x,y\}$ and $i \in \{2,3,\dots,\binom{p}{2}\}$. Since this obviously only holds when $|A| \leq 2$ we have our final contradiction which includes this case.

■

At the end of this epistle about consequences of decisiveness, we only have to prove a theorem about the interference between two such decisive coalitions. This theorem is similar to (4.3.7).

<u>Lemma 4.4.25</u>

Assume (4.4.19), $|A| \geq 3$, $S,T,M \in 2^N$, with $S \cap T \cap M = \varphi$ and $S \cup T \cup M = N$.

Then it cannot be the case that, for all $X \in \{S,T,M\}$ and $R \in L(A)$: $F(\langle R^X, \bar{v}R^{N-X}\rangle) = R$.

<u>Proof of lemma 4.4.25</u>

Suppose for all $X \in \{S,T,M\}$ and for all $R \in L(A)$: $F(\langle R^X, \bar{v}R^{N-X}\rangle) = R$.

Let $A = \{a,b,x,d_1,d_2,\dots,d_k\}$, where $k = p-3$.

It will be proven that the profile $r \in L_n(A)$ has no image, hence we have a contradiction. Here r is defined as follows:

$abcd_1d_2.\dots.d_k : R^i$ for all $i \in S \cap T$,

$bcad_1d_2.\dots.d_k : R^i$ for all $i \in T \cap M$, and

$cabd_1d_2.\dots.d_k : R^i$ for all $i \in S \cap M$.

Let $r_1 \in L_n(A)$ be defined as follows:

$abcd_1d_2.\dots.d_k : R_1^i$ for all $i \in S \cap T$,

$abcd_1d_2.\dots.d_k : R_1^i$ for all $i \in T \cap M$, and

$cabd_1d_2.\dots.d_k : R_1^i$ for all $i \in S \cap M$.

Then by the foregoing lemma's and the decisiveness assumption of T it follows that $abcd_1d_2.\dots.d_k : F(r_1)$.

Furthermore, $\tilde{l}(r_1,r) = 2$, hence $\tilde{c}(F(r_1),F(r)) \le 2$.
Take r_2: $bacd_1d_2.....d_k : R^i$ for all $i \in S \cap T$,
$bacd_1d_2.....d_k : R^i$ for all $i \in T \cap M$, and
$cabd_1d_2.....d_k : R^i$ for all $i \in S \cap M$.
Similarly it follows that $bacd_1d_2.....d_k : F(r_2)$ and $\tilde{c}(F(r_2),F(r)) \le 2$.
By similar reasonings using the decisiveness of S and M, there are $r_3,r_4,r_5,r_6 \in L_n(A)$, with $\tilde{c}(F(r),F(r_i)) \le 2$, for all $i \in \{3,4,5,6\}$ and $acbd_1d_2.....d_k : F(r_3)$, $cabd_1d_2.....d_k : F(r_4)$, $bcad_1d_2.....d_k : F(r_5)$, and $cbad_1d_2.....d_k : F(r_6)$.
Take $\bar{R}_i = F(r_i)|_{\{a,b,c\}}$ for $i \in \{1,..6\}$ and $\bar{R} = F(r)|_{\{a,b,c\}}$.
Obviously $\{\bar{R}_1, \bar{R}_2,...\bar{R}_6\} = L(\{a,b,c\})$, $\bar{R} \in Q(\{a,b,c\})$, and $\tilde{c}(\bar{R},\bar{R}_i) \le 2$ for all $i \in \{1,2,..6\}$. Since there evidently is a $R \in L(\{a,b,c\})$, with $\bar{a}\bar{R} \subseteq R$ this is impossible.

■

Now we can prove some impossibility results.

Theorem 4.4.26

Suppose $\Gamma = \langle A,N \rangle$ is a society, with $|A| \ge 3$ and $N = \{1,2,..,.n\}$, V is a classified set of orderings and F is a welfare function from $L_n(A)$ to $V(A) \subseteq Q(A)$ on Γ, with $\{R \cup \bar{R} : R,\bar{R} \in L(A)\} \not\subseteq V(A)$.
Then F is not simultaneously, Pareto-optimal, not strongly dictatorial and $\langle \tilde{l}_n,\tilde{c} \rangle$-non-expansive.

Proof of theorem 4.4.26

Let all the variables be as above and suppose F is non-dictatorial, Pareto-optimal and $\langle \tilde{l}_n,\tilde{c} \rangle$-non-expansive.
By the assumption that $V(A) \not\subseteq \{R \cup \bar{R} : R,\bar{R} \in L(A)\}$ it follows from lemma 4.4.23 that for all $S \subseteq N$ there is a $R \in L(A)$, such that $F(\langle R^S,\bar{v}R^{N-S} \rangle) \neq A \times A$.
Using lemma 4.4.21, 4.4.22 and 4.4.24 we have that for all $S \subseteq N$ there is a $X \in \{S, N-S\}$, such that for all $r \in L_n(A)$ and all $R \in L(A)$:
if $R^i = R$, for all $i \in X$, then $F(r) = R$. (4.4.26.1)
By the non-dictatorship of F it follows that for all $i \in N$

all $N_i := N-\{i\}$ and all $R \in L(A)$: $F(<R^{\{i\}}, \bar{v}R^{N-\{i\}}>) = \bar{v}R$.
Since N is finite there is a smallest $S \subseteq N$ satisfying (4.4.26.1).
If $|S| = 1$, we have our dictator, and therefor a contradiction.
If $|S| \geq 2$. Take $i \in S$. By (4.4.25) $S - \{i\}$ has again property (4.4.26.1) (use S,N_i). Hence, we have a contradiction on the minimality of S.

■

This easily leads to the following result.

Corollary 4.4.27

Suppose $\Gamma = <A,N>$ is a society, with $|A| = p$ and $|N| = n$, F is a Pareto-optimal and $<1_n,c>$-non-expansive welfare function from $L_n(A)$ to $V(A) \subseteq Q(A)$ on Γ, where V is classified as a set of orderings.
Then F is strongly dictatorial if one of the following is the case:

4.4.27.1 $3 \leq |A| = p \leq 5$ and $V(A) \subset Q(A)$,
4.4.27.2 $|A| = p \geq 3$ and $V(A) \subseteq W(A)$,
4.4.27.3 $|A| = p \geq 3$ and $A \times A \notin V(A)$, or
4.4.27.4 $|A| = p \geq 4$ and $V(A) \subseteq I(A)$.

Proof of corollary 4.4.27

(4.4.27.1) Note that if $|A| = p$, $p \in \{3,4,5\}$, then $\{R \cup \bar{R} : R,\bar{R} \in L(A)\} = Q(A)$. Hence, in that case we are trivialy done by (4.4.26).
(4.4.27.2) If $V(A) \subseteq W(A)$, then obviously $\{R \cup \bar{R} : R,\bar{R} \in L(A)\} \not\subseteq V(A)$ and we can use (4.4.26) straightforwardly.
(4.4.27.3) and (4.4.27.4) are similar to (4.4.27.2).

■

We finish this sequence of impossibilities with one in which the domain is not restricted to $L_n(A)$.

Theorem 4.4.28

Let $\Gamma = \langle A,N \rangle$ be a society, with $|A| \geq 3$ and $|N| = n$. Let F be a welfare function on Γ from $W_n(A)$ to $V(A)$, where V and W are classified sets of orderings, with $\{R \cup \bar{R} : R,\bar{R} \in L(A)\} \not\subseteq V(A)$, $EC(W_n(A))$ is connected and $W(A),V(A) \subseteq Q(A)$.

Then F is not simultaneously Pareto-optimal, $\langle \tilde{q}_n, \tilde{c} \rangle$-non-expansive and not strongly-semi-dictatorial. Where F is strongly-semi-dictatorial, iff there is an $i \in N$ such that for all $r \in W_n(A)$: if $R^i \in L(A)$, then $F(r) = R^i$.

Proof of theorem 4.4.28

Obviously $F|_{L_n(A)}$ is dictatorial.

Observing that in (4.4.24) essentially only the connectedness of $L_n(A)$ is used, it becomes clear that the dictator of $F|_{L_n(A)}$ is a semi-dictator of F.

■

Evidently the other non-expansiveness criteria lead also to impossibilities. However, one has to take care of the correspondence between the domain of the distance functions and the (co)domain of the welfare functions. The obvious idea that non-expansiveness is a weaker condition than the independence of irrelevant alternatives needs some formal proof. This is done by the following example:

Example 4.4.29

In this example we will define a strongly Pareto-optimal, neutral, strongly positively associated, anonymous, $\langle \tilde{c}_n, \tilde{q} \rangle$-non-expansive and not weakly dictatorial welfare function F on $\Gamma = \langle A,N \rangle$, where $|A| \geq 3$ and $|N| = n \geq 4$, from $L_n(A)$ to $Q(A)$. The proof that this F has all the properties stated above is in Storcken [1988]. The reader may verify this.

Let $\Gamma = \langle A,N \rangle$, $|A| \geq 3$ and $|N| = n \geq 4$. We will first define some classes of profiles in $L_n(A)$ on which coalitions have some decisive power. These classes are far from each other in the $\tilde{l}$-distance sense. Hence, there is no interference of these decisiveness powers. Let $V_{<a,b>}$ for $a,b \in A$, $a \neq b$, be defined as follows:

$V_{<a,b>}$:= {$r \in L_n(A)$: there is a numbering x_3 up to x_p of $A - \{a,b\}$, there are $R_1,R_2,R_3,R_4 \in L(A)$ and four coalitions $S_1,S_2,S_3,S_4 \subseteq N$ which form a partition of N, such that:

(1) $|S_2| = |S_3| = |S_4| = 1$.

(2) for all $i \in N$: $R_t = R^i$, iff $i \in S_t$, where

$abx_3x_4.......x_p$: R_1,

$x_px_{p-1}...x_4x_3ab$: R_2,

$ax_3x_5x_7..bx_4x_6..$: R_3, and

$...x_8x_6x_4b..x_{11}x_9x_3x_5a$: R_4}.

Define $F : L_n(A) \to Q(A)$ as follows

$$F(r) := \begin{cases} \bar{c}\{<b,a>\}, \text{ iff } r \in V_{<a,b>} \text{ for some } a,b \in A. \\ \cup \{R^i : i \in N\}, \text{ iff } r \notin V_{<a,b>} \text{ for all } a,b \in A. \end{cases}$$

Then it is straightforward, but cumbersome and technical, to prove that F is a well-defined, strongly Pareto-optimal, strongly positively associated, $<l_n,q>$-non-expansive, neutral, anonymous and not weakly dictatorial.

■

This example makes two things clear:

(1) The independence condition is stronger than the non-expansiveness condition, even under very strong other assumption imposed on a welfare function.

(2) The conditions of (4.4.26), (4.4.27), (4.4.28) about the range of F are necessary.

Because the independence of irrelevant alternatives condition implies non-expansiveness and not the other way arround, it is clear that the above stated theorems are stronger than well-known impossibilities as discussed in § 4.3.

In literature this insight is already developing (See Chichilnisky & Heal [1983], Chichilnisky [1969, 1980 and 1982], Visser [1988], and Baigent [1985 and 1987]). However, the results presented here are not comparable to those in literature, since other conditions are imposed on the welfare function. Instead of non-dictatorship, anonymity is imposed. Furthermore, in almost all works (except for Baigent [1987] and Visser [1988]) $|A|$ is not finite. Moreover, in Baigent [1987] as well as in the others the independence of irrelevant alternatives is replaced by continuity (when $|A|$ is not finite) or proximity-preservation (See (4.4.3.1.)). Although at first sight non-expansiveness seems to be stronger than continuity, it is obvious from chapter 3 that non-expansiveness as used here is in fact a continuity property. However, it is not clear which property is the stronger one: non-expansiveness for discrete metric spaces or continuity for `infinite' spaces, because they are not comparable. Even their topological spaces are not comparable. But we can compare proximity-preservation and non-expansiveness.

In general it is still an open question whether a specific interpretation of proximity-preservation implies a specific non-expansiveness condition. On the other hand, the independence condition does not imply it, which is shown by the following example. Hence, several non-expansiveness conditions do not imply this proximity conditon.

<u>Example 4.4.30</u>

Let F be a welfare function on society $\langle A,N\rangle$, such that $p = |A| \geq 3$, $|N| = n$ and $A = \{x,y,a_3,a_4,\ldots,a_p\}$, from $W_n(A)$ to $W(A)$, defined as follows:

$$F(r) = \begin{cases} R_{xy} & \text{iff } r \in W_n(A,\ x > y) \\ R_{yx} & \text{iff } r \in W_n(A) - W_n(A,\ x > y), \end{cases}$$

where $W_n(A,\ x > y) = \{r \in W_n(A)\text{: for all } i \in N,\ x > y : R^i\}$, $xya_3a_4\ldots a_p : R_{xy}$ and $yxa_3a_4\ldots a_p : R_{yx}$.

Obviously F is $\langle \tilde{q}_n,\tilde{q}\rangle$-non-expansive, independent of irrelevant alternatives, anonymous, non-dictatorial. But F is neither Pareto-optimal nor proximity-preserving for all its interpretations.

The former is trivial, the last is shown now.

Take $r_1 = \langle R_{xy},R^2,R^3,\ldots,R^n\rangle$, $r_2 = \langle R^1_2,R^2,R^3,\ldots,R^n\rangle$, $r_3 = \langle R_{yx},R^2,R^3,\ldots,R^n\rangle$, such that $R^2,R^3,\ldots R^n \in W(A,\ x > y)$ and $a_3xya_4a_5\ldots a_p : R^1_2$.

Then $d_{\tilde{q}}(r_1,r_2) > d_{\tilde{q}}(r_1,r_3)$.

But $\tilde{q}(F(r_1),F(r_2)) < \tilde{q}(F(r_1),F(r_3))$. Hence, F is not proximity-preserving.

■

Example 4.4.30 shows that proximity preservation is not a weaker condition than the independence of irrelevant alternatives or a non-expansiveness conditions. Since the F of (4.4.30) is anonymous this holds even under anonymous functions. Clearly, if F is Pareto-optimal we have dictatorship, which evidently preserves proximity.

This is also done since we fail in giving a correct interpretation of the proximity-preservation. At first sight, we are willing, to interpret proximity-preservation as Baigent [1987] by `small errors in establishing the individual preferences only lead to small errors in establishing the social preference'. However, after rereading Baigent's illustration (page 163 : the only place where proximity-preservation is correctly formulated), we formulate this concept as follows: A "greater" error in establishing individual preferences only leads to a "greater" or equally "great" error in establishing the social preference. This formulation reveals that a proximation preservation is a globally universally quantified condition, where continuity is a locally universally quantified condition. It cannot be said, whether proximity-preservation is strongly related to continuity. This can be said of non-expansiveness (See chapter 3). There is doubt about the proximity-preservation being a `weak' condition. This can be illustrated by the following intermezzo, where under mild conditions proximity-preservation and Pareto-optimality conflict with each other. Hence, appart from any other social condition Pareto-optimality and proximity-preservation lead already to a contradiction.

<u>Intermezzo</u>

Let $\Gamma = \langle A,N \rangle$ be a society, such that $|A| = p$, $|N| = n$ and $3 \leq n \leq p!$.

Furthermore, let V be classified as a set of orderings and δ a distance function on $V(A)$, such that $\langle V(A),\delta \rangle$ is a full metric space.

Suppose F is a welfare function from $V_n(A)$ to $V(A)$ which is proximity-preservative with respect to δ.

Then F is constant, i.e. $|F(V_n(A))| = 1$.

Here proximity preserving is that for all $r_1, r_2, r_3 \in V_n(A)$, with $d_\delta(r_1,r_2) \le d_\delta(r_1,r_3)$: $\delta(F(r_1),F(r_2)) \le \delta(F(r_1),F(r_3))$.

$(d_\delta(r_1,r_2) := \sum_{i=1}^{n} \delta(R_1^i,R_2^i))$.

<u>Proof of this assertion</u>

Without loss os generality let $\text{mesh}(V(A),\delta) = 1$.

Hence, $\langle V_n(A), d_\delta\rangle$ is full and has meshwidth 1.

Define $\langle R_1,R_2\rangle \in E :\Leftrightarrow R_1,R_2 \in V(A)$ and $\delta(R_1,R_2) = 1$

$\langle r_1,r_2\rangle \in E_n :\Leftrightarrow r_1,r_2 \in V_n(A)$ and $\delta(r_1,r_2) = 1$.

Let $k = |F(V_n(A))|$.

Suppose $k > 1$. We deduce a contradiction.

Obviously $k \le |V(A)|$.

Let $F(r_1) = R_1$ and $F(r_2) = R_2$, such that $\langle r_1,r_2\rangle \in E_n$ and $R_1 \ne R_2$.

Take $r_3 \in F^{-1}(R_1)$.

If $d_\delta(r_3,r_1) \ge 2$, then $d_\delta(r_3,r_1) > d_\delta(r_2,r_1)$ but

$0 = \delta(F(r_1),F(r_3)) < \delta(F(r_1),F(r_2))$.

Hence, $\langle r_1,r_3\rangle \in E_n$.

Since $\langle V_n(A),d_\delta\rangle$ is full, hence E_n is connected we have just proven that:

For all $R \in F(V_n(A))$ there is a $r \in V_n(A)$, such that for all $\hat{r} \in F^{-1}(R)$ $\langle r,\hat{r}\rangle \in E_n$.

Hence, for all $R \in F(V_n(A))$, there is a $r \in V_n(A)$, such that $F^{-1}(R) \subset B(r,1\tfrac{1}{2},V_n(A),d_\delta)$.

How many elements are there in such a ball $B(r,1\tfrac{1}{2},V_n(A),d_\delta)$?

If $\hat{r} \in B(r,1\tfrac{1}{2},V_n(A),d_\delta)$, it can only differ from r at precisely 1 coordinate.

Hence, $|B(r,1\tfrac{1}{2},V_n(A),d_\delta)| \le n \cdot |V(A)|$.

So $|F^{-1}(F(V_n(A)))| < k \cdot n \cdot |V(A)| \le |V(A)|^2 \cdot n$.

Since $V_n(A) = F^{-1}(F(V_n(A)))$ and $L(A) \subseteq V(A)$ it follows:

$$|V(A)|^n = |V_n(A)| = |F^{-1}(F(V_n(A)))| < n \cdot |V(A)|^2.$$

Hence, $n > |V(A)| \ge |L(A)| = p!$

This contradicts our assumption.

■

Let us show the use of the corollary 3.5.11 to other types of impossibility theorems in social choice theory. This is done by virtue of two examples. It is not our intention to give a full proof of some impossibility results as is done in the deduction of (4.4.26), (4.4.27) and (4.4.28). It is only pointed out here that these other impossibility results exist, i.e. they are proven up to the level of (4.4.21) in the proof of theorem 4.4.26, the rest is left to the reader.

<u>Example 4.4.31</u> <u>Impossiblities and intensities</u>

One of the most important criticisms on the models presented in this chapter about orderings is, that no intensity of a preference between two alternatives is taken into acount. We will deal with these intensities here and will show that they will not lead automatically to a possibility theorem. So this criticism is not substantional for the impossibilities, that is, if we interprete intensities in the following way.

Let $|A| = p$, $A = \{a_1,a_2,\ldots,a_p\}$, $\binom{p}{2} = k + 1$ and f a bijection from $\{\{a_i,a_j\} : a_ia_j \in A, i \neq j\}$ to $\{0,1,2,\ldots,k\}$.

Let $x = \langle x_0,x_1,\ldots,x_k\rangle \in R^{k+1}$.

x is interpreted as the complete relation on A, such that if $f(\{a_i,a_j\}) = t$ and $i < j$, then

if $x_t = 0$, then a_i and a_j are indifferent,

if $x_t > 0$, then a_i is strictly prefered to a_j with intensity x_t, and

if $x_t < 0$, then a_j is strictly prefered to a_i with intensity x_t.

<u>Assumption 4.4.31.1</u> For all $\alpha > 0$: x and $\alpha \cdot x$ interprete the same complete relation on A.

This assumption means that only the ratio's of intensity matter. In essence we therefore have that all relations are in $\{\langle 0,0,0,\ldots,0\rangle\} \cup S^k$, where $S^k = \{x \in R^{k+1} \mid ||x|| = 1\}$

<u>Assumption 4.4.31.2</u> Total indifferences are excluded.

By these two assumptions all the essential information is in S^k. Hence, S^k represents the set of complete relation on A with intensities except the total indifference. A <u>welfare function</u> for a society $\langle A,\{1,2\}\rangle$ is now a function F from $S^k \times S^k$ to S_k.

<u>Assumption 4.4.31.3</u> F is unanimity respecting, i.e. for all $x \in S^k$, $F(\langle x,x\rangle) = x$.

<u>Assumption 4.4.31.4</u> F is $\langle d_2,d\rangle$-non-expansive, where $d(x,y) := \arccos(x \cdot y)$ for $x,y \in S^k$ (it is the arc-distance between x and y) and $d_2(\langle x^1,y^1\rangle,\langle x^2,y^2\rangle) := d(x^1,x^2) + d(y^1,y^2)$.

Note that these assumptions are very mild. For instance there is no assumption on the transtivity of x or its intensity.

Now we will prove a result resembling (4.4.21).

<u>Claim 4.4.31.5</u> Either for all $x \in S^k$, $F(\langle x,-x\rangle) = x$, or for all $x \in S^k$, $F(\langle x,-x\rangle) = -x$.

Observe that x and -x interpret relations which are conversions of each other. Hence, by claim 4.4.31.5 it follows that either individual 1 wins completely in every maximal conflict or individual 2 does.

Obviously by the non-expansiveness it is sufficient to prove:

<u>Claim 4.4.31.6</u> For all $x \in S^k : F(\langle x,-x\rangle) \in \{x,-x\}$.

<u>Proof of claim 4.4.31.6</u>

Without loss of generality suppose $x = \langle 0,0,0,\ldots,0,1\rangle$.

Take for all $0 \le t < k$

$L_t' = \{y \in S^k : y = \langle y_0,\ldots,y_k\rangle \text{ and } y_t \le 0\}$ and

$R_t' = \{y \in S^k : y = \langle y_0,\ldots,y_k\rangle \text{ and } y_t \ge 0\}$.

Now $NH_M(L_t',R_t') = \{y \in S^k : y = \langle y_0,\ldots,y_k\rangle \text{ and } y_t = 0\}$, where $M = \langle S^k,d\rangle$.

To prove (4.4.31.6) it is sufficient to prove $F(\langle x,-x\rangle) \in NH_M(L_t',R_t')$, for all $t \ge 0$, $t < k$.

Without loss of generality it is sufficient to prove this for t = 0. This is done by virtue of (3.5.11). Again because of similarities it is sufficient to prove that (4.4.31.7) and (4.4.31.8) hold. Where

4.4.31.7 R_0' circularly encloses R_0 with diameter π, and

4.4.31.8 R_0 can be approximated by interiors of ellipses from x and -x and radius π.

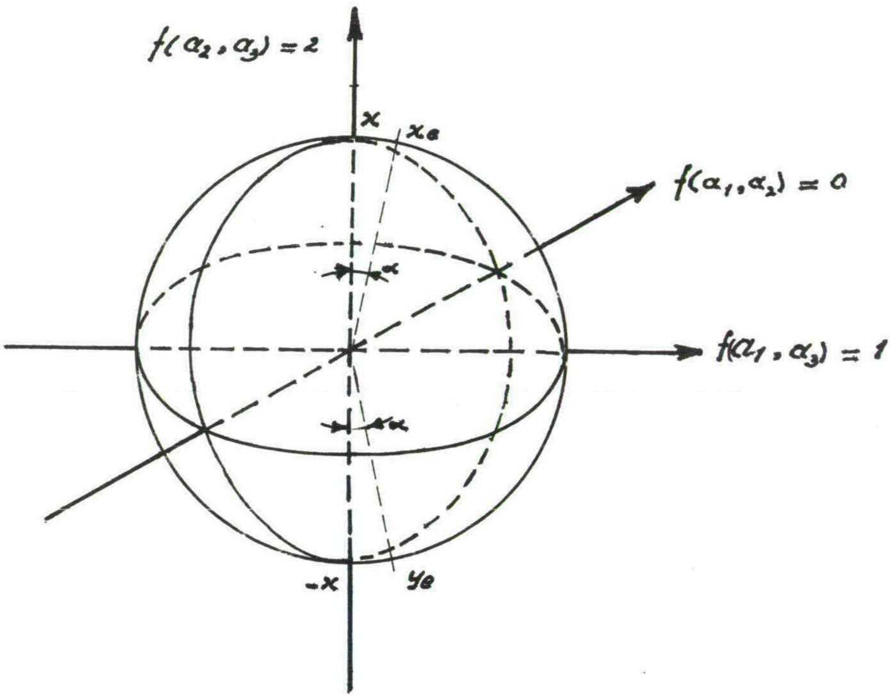
Here $R_0 = \{y \in S^k : y = \langle y_0,\ldots,y_k\rangle$ and $y_0 > 0\}$.

(4.4.31.7) is evident.

(4.4.31.8) Take $e > 0$ it is sufficient to prove that there are x_e, $y_e \in S^k$, with $d(x,x_e) + d(-x,y_e) < e$ and, for all $z \in L_0$, $d(x_e,z) + d(y_e,z) < \pi$.

Take $\alpha < \frac{1}{4}e$. $x_e = \langle +\sin\alpha,0,0,\ldots,0,\cos\alpha\rangle$ and $y_e = \langle +\sin\alpha,0,0,\ldots,0,-\cos\alpha\rangle$.

Obviously we have the following picture for $1 + k = 3$.



Now $x_e \cdot x = \cos\alpha = y_e \cdot -x$. Hence, $d(x_e,x) = d(y_e,-x) = \alpha$.

Hence, $d(x_e,x) + d(-x,y_e) < e$.

Take $z \in R_0$. $z = \langle z_0,z_1,\ldots,z_k\rangle$, where $z_0 > 0$.

We have to prove that: $d(x_e,z) + d(y_e,z) < \pi$.

Hence, we have to prove that:

$\arccos(z_0\cdot\sin\alpha + z_k\cdot\cos\alpha) + \arccos(z_0\cdot\sin\alpha - z_k\cdot\cos\alpha) < \pi$.

arccos is strictly decreasing $z_0 > 0$, α is very small, so

$\cos\alpha > 0$ and $\sin\alpha > 0$. Moreover
$\arccos(z_0 \cdot \sin\alpha + z_k \cdot \cos\alpha) + \arccos(z_0 \cdot \sin\alpha - z_k \cdot \cos\alpha) <$
$\arccos(z_k \cdot \cos\alpha) + \arccos(-z_k \cdot \cos\alpha) = \pi$.
Which completes the proof of claim 4.4.31.6. To complete the impossibility it would be necessary to extend this decisiveness condition further. The author did not investigate this fully but it is very likely that this is the case, because of some results already deduced.

■

In the previous example an impossibility was deduced for welfare functions on orderings with intensities. In our last example, which should again demonstrate the value of (3.5.11) we are deducing an impossibility theorem for choice correspondences based on individual choices.

<u>Example 4.4.32</u> <u>Choice correspondences based on individual choices</u>.

Let $\langle A,\{1,2\}\rangle$ be again a society such that $|A| \geq 3$. A choice correspondence C based on individual choices is a function from $2^A \times 2^A$ to 2^A.

To every combination of individual choices $\langle B^1,B^2\rangle \in 2^A \times 2^A$. C assings a collective choice $C(\langle B^1,B^2\rangle) \in 2^A$.

<u>Assumption 4.4.32.1</u> C is unanimous, i.e., for all $B \in 2^A$:

$C(\langle B,B\rangle) = B$.

<u>Assumption 4.4.32.2</u> C is $\langle d_2,d\rangle$-non-expansive, where
$d_2(\langle B_1^1,B_1^2\rangle,\langle B_2^1,B_2^2\rangle) := d(B_1^1,B_2^1\rangle + d(B_1^2,B_2^2)$ and $d(X,Y) = |X \triangle Y|$.

Take $M = \langle 2^A,d\rangle$,

<u>Claim 4.4.32.3</u> For all $B \in 2^A - \{A,\varphi\}$: $C(B,A-B) \in \{B,A-B\}$.

If claim 4.4.32.3 is proved, we have again an impossibility.

Let $R_{xy} = \{B \in 2^A : x \in B \text{ and } y \in B\}$.

$L_{xy} = \{B \in 2^A : x \notin B \text{ and } y \notin B\}$.

$R'_{xy} = \{B \in 2^A : x \in B \text{ or } y \in B\}$.

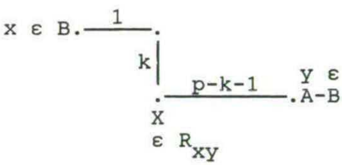
$L'_{xy} = \{B \in 2^A : x \notin B \text{ or } y \notin B\}$.

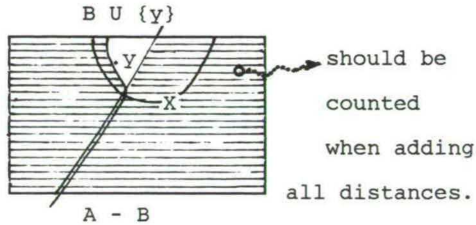
It is obvious that:

$B \in NH_M(L_{xy},R_{xy}) \Leftrightarrow (x \in B \;\&\; y \notin B) \vee (x \notin B \;\&\; y \in B)$.

Moreover, R'_{xy} circularly encloses R_{xy} and L'_{xy} circularly

encloses R_{xy} both with diameter p. This is again straightforwardly to prove.
To prove, that e.g. R_{xy} (can be approximated by interiors of ellipses from B and A-B and radius p, where $x \varepsilon B$ and $y \varepsilon A-B$, note that the following picture holds for any $X \varepsilon R_{xy}$.





Similarly L_{xy} can be approximated from B and B-A and radius p.
Hence, by (3.5.11) it obviously holds that:
For all $x \varepsilon B$ and $y \varepsilon A-B$: $F(\langle B,A-B\rangle) \varepsilon NH_M(L_{xy},R_{xy})$.
Now since for $x \varepsilon B$ and $y \varepsilon A-B$:
$x \varepsilon F(\langle B,A-B\rangle)$ and $y \notin F(\langle B,A-B\rangle)$ or
$x \notin F(\langle B,A-B\rangle)$ and $y \varepsilon F(\langle B,A-B\rangle)$.
Using $|A| \geq 3$ it follows that $F(\langle B,A-B\rangle) \varepsilon \{B,A-B\}$.
Furthermore:

<u>Claim 4.4.32.4</u> Either for all $B \varepsilon 2^A - \{A,\varphi\}$, $F(\langle B,A-B\rangle) = B$, or for all $B \varepsilon 2^A - \{A,\varphi\}$ $F(\langle B,A-B\rangle) = A-B$.
This makes the impossibility again obvious.

■

Although in the foregoing examples the collective constitutional rules were not (yet) proved to be dictotorial and the society only had two individuals, they still gave an insight in the power of corollary 3.5.11. It is just a question of time and paper to strengthen these results such that the rules discussed above appear to be dictatorial. Because of the effort which those proofs certainly would involve, compared to the small increment of insight, we did not investigate this further.

In this section 4.4 we have seen that a meaningful substitute for the independence of irrelevant alternatives in order to avoid impossibilities is hard to find. The welfare functions constructed in example 4.4.29 are hard to use in practice, since it is not easy to determine the collective preference at a given profile. Furthermore, the rule does not often yield a decision, mostly these occur on a unanimity basis.

Moreover, it has become apparant in the proofs of the theorems of this section, that the maximal conflict situation between a coalition and its complement gives rise to an impossibility. That is a consequence of Arrow's formal model, as developed and designed until now. The results based on the outcomes at maximal conflict situations could be interpreted as follows: In those situations of maximal conflict there is no resemblance between the conflicting parties; they act as if they are completely independent from each other, therefore the central decision maker F cannot act in these situations properly and yields such strange results.

Anymore "realistic" result requires further elaboration and improvements of the axioms on collective choice. For instance, one might doubt the assumption that all such conflicts occur and assume that the preferences between individuals influence each other. So the domain is restricted. In the following section, we investigate, what happens when the unrestricted domain condition is dropped. Hence, not all those odd profiles initiating intuitively impossibilities are admitted.

§ 4.5 Restricted Domains

In this section we study some effects on the existence of welfare functions, caused by restrictions on the set of possible profiles. It is interesting to know whether or not a given welfare functions, satisfying specific social properties, can be constructed on a given subset of the set of profiles. This general question is difficult to answer, even if the type of welfare functions is fixed. Therefore, we will take a fixed type of welfare functions and also special types of subsets of the set of profiles. For these special restrictions we will give necessary and sufficient conditions for the restricted set of profiles, such that on this domain there exists (in a constructive way) a welfare function of that special chosen type.

In literature this is called the restricted domain approach. A great amount of work in this field is devoted to an investigation of necessary and sufficient conditions for the set of profiles in order that there exists a welfare function, which determines the preferences on the basis of a simple majority decision. First Condorcet [1785] studied such decision rules, May [1952] characterized simple majority rules for 2-alternative sets. Black [1948] and Arrow [1978] introduced a sufficient condition named `single peakness'. Several other authors, e.g., Inada [1964], formulated other sufficient conditions. Finally Pattanaik & Sen [1969] found necessary and sufficient conditions for the set of profiles such that a simple majority rule exists. After this, several weakenings of the majority principle are studied see, e.g., Fishburn [1970], Chichilnisky and Heal [1983] and Monjardet [1979]. For more information see also Sen [1986].

In all theorems introduced above, the necessary and sufficient conditions for the existence of a majority rule, typically exclude profiles. Hence, the individuals are not supposed to choose their individual ordering independently from each other. Here we will suppose that every individual has a set of possible orderings, which can occur independently from those of the other orderings of the other individuals. Moreover, not the majority principle is imposed on a welfare function, but the conditions positive association, non-dictatorship and strong

Pareto-optimality. Although the majority principle implies these properties, the reverse is not true. Of course, welfare functions imposed by the conditions as proposed here, resemble very much those, which are imposed with the simple majority principle (See e.g. May [1952] and Monjardet [1979]). In literature this type of study can also be found in e.g. Kalai & Muller [1977], Kalai & Ritz [1980], Maskin [1976] and Ritz [1985].

Let us start with some notational conventions. Suppose A is a set (finite) of alternatives, N a set of individuals $N = \{1,2,\ldots,n\}$ and V a set of classified orderings. We will suppose that each individual $i \in N$ has a set of possible individual orderings on A: $V(A)_i \subseteq V(A)$. In that case the set of possible combinations of individual orderings is the following set of n-tuples: $\{\langle R^1,R^2,\ldots,R^n\rangle$: for all $i \in N$ $R^i \in V(A)_i\}$. Again the subscripts of the domain of a relation is dropped. This set will be indicated by $V(A)_N$. Of course it is possible that $V(A)_i = V(A)$ for all $i \in N$ and hence $V_n(A) = V(A)_N$. Note that the place of the indexation is essential. Again we have the notion welfare function and its conditions. Formally it would be necessary to redefine all these notions for the restricted case here. Since it is evident how to do this, it is left to the reader.

The first type of restrictions is the following. Only the set of orderings of one individual is restricted. Hence, we are investigating situations, in which $V(A)_i = V(A)$ for all $i \geq 2$ and $V(A)_1 \subseteq V(A)$. In some sense this type of restrictions is mimimal, only 1 set of individual orderings is restricted, therefore the domain is maximal in that sense. It is called the <u>maximal-domain-approach</u>. Actually this maximal-domain-approach is a translation of the results found there.

First the necessary and sufficient conditions for this maximal-domain-approach are introduced.

Definition 4.5.1 Inseparable pair - Inseparable set

Let A be a set of alternatives, $x,y \in A$, $B \subseteq A$ and $V(A) \subseteq L(A)$ a set of linear orderings on A. Then

4.5.1.1 $\langle x,y\rangle$ is an inseparable pair of V(A), iff $V(A,x>y) \neq \varphi$ and for all $R \in V(A,x>y)$, $x \sim y$: sucR, and

4.5.1.2 B is an inseparable set of V(A), iff for all $R \in V(A)$ and all $a,b \in B$ and $c \in A$: if $a > c > b : R$, then $c \in B$.

■

Inseparable pairs have already been discussed in literature, see e.g. Kalai & Ritz [1980] and Ritz [1985]. B is an inseparable set of V(A), iff all the preferences between pairs in B cannot be seperated by an element in A-B in any relation of V(A). Obviously there are trivial separable sets B, such as A = B, $|B| = 1$, or φ. Let B be a separable set of $V(A) \subseteq L(A)$.
Let $R \in V(A)$. Take $R|_B$, $R|_{((A-B) \cup \{b\})}$, with $b \in B$, $\sigma \in S_U$, with $\sigma(b) = x$, $\sigma(x) = b$, $x \notin A$, and for all $y \in U - \{x,b\}$, $\sigma(y) = y$. Then $\sigma R|_{((A-B)\cup\{b\})} \in L((A-B) \cup \{x\})$ and $R = Sub(\sigma R|_{(A-B)\cup\{b\}}, x, R|_B)$. If B is a separable set of V(A) then the set B is ordered in every relation of V(A) in a cluster. See also Blau [1957], who used this notion only intuitively.

We will prove the following theorem.

Theorem 4.5.2 Maximal domain characterization

Suppose $\Gamma = \langle A,N\rangle$ is a society, $i \in N$, $|N| \geq 2$ and for all $j \in N - \{i\}$: $L(A)_j = L(A)$.

Then (4.5.2.1) and (4.5.2.2) are equivalent.

(4.5.2.1) There is a welfare function F on Γ from $L(A)_N$ to W(A), which is simultaneously strongly Pareto-optimal, positively associated and strongly non-dictatorial.

(4.5.2.2) There is an inseparable pair $\langle x,y\rangle$ of $L(A)_i$ or there is an inseparable set B of $L(A)_i$, with $|A| > |B| > 1$.

■

The proof of this theorem requires several lemmas. First we prove (4.5.2.2) -> (4.5.2.1) by the following two lemmas.

Lemma 4.5.3

Suppose $\Gamma = \langle A,N \rangle$, $|A| \geq 3$, $N \geq 2$, $L(A)_i = L(A)$ for all $i \geq 2$ and $x,y \in A$, such that $\langle x,y \rangle$ is inseparable for $L(A)_1$. Then (4.5.2.1).

Proof of lemma 4.5.3

Although the proof is known in literature (See e.g. Ritz [1985]), we give a proof here to have a full `stand-alone' proof of theorem 4.5.2 and to demonstrate the advantage of our notations.

For all $r \in L(A)_N$ define

$$F(r) = \begin{cases} R^1, \text{ iff } R^1 \in L(A,y>x) \\ (R^1 \cap E_{xy}) \cup (\{x,y\} \times \{x,y\} \cap R^2), \text{ iff } \\ \qquad R^1 \in L(A,x>y). \end{cases}$$

Note that if $R^1 \in L(A,x>y)$, then $\langle F(r),R^1 \rangle \in EC(Q(A),\{x,y\})$ because of the inseparability of $\langle x,y \rangle$ of $L(A)_1$.

Hence, $F : L(A)_N \to L(A)$.

All the social conditions can straightforwardly be proved.

■

Lemma 4.5.4

Suppose $\Gamma = \langle A,N \rangle$, $|A| \geq 3$, $|N| \geq 2$, $L(A)_i = L(A)$ for all $i \geq 2$ and B is an inseparable set of $L(A)_1$, with $1 < |B| < |A|$. Then (4.5.2.1).

Proof of lemma 4.5.4

Again we will define an appropriate welfare function. Take $\sigma \in S_U$ such that $\sigma(b) = x$, $\sigma(x) = b$, and $\sigma(z) = z$ for all z / $\{x,b\}$, where $x \notin A$ and $b \in B$.

Define F for all $r \in L(A)_N$ as follows:

$$F(r) := Sub((\sigma R^1)|_{(A \cup \{x\}) - B}, x, R^2|_B).$$

Since L(U) can be classified as a set of orderings, and is substitutionally closed over itself it follows $F(r) \in L(U)$. Furthermore, x is substituted by $R^2|_B$. Hence, $F(r) \in L(A)$.

Hence, F is a welfare function from $L(A)_N$ to $L(A)$.

The individuals in $\{1,3,....,n\}$ are no weak dictators, since $F(r)|_B = R^2|_B$ and $|B| \geq 2$. Individual 2 is not a weak dictator, since $F(r)|_{\{a,b\}} = R^1|_{\{a,b\}}$ for all $a \in A - B \neq \varphi$. All the other conditions follow similarly simple.

■

We will now prove theorem 4.5.2.

Proof of theorem 4.5.2

(4.5.2.2) -> (4.5.2.1) Follows from lemma 4.5.3 and 4.5.4.

(4.5.2.1) -> (4.5.2.2) Suppose $F' : L(A)_N \rightarrow W(A)$ is a strongly non-dictatorial, strongly Pareto-optimal and positively associated welfare function on Γ.

Define $F : L(A)_N \underset{\sim}{\rightarrow} L(A)$ as follows.

$F(r) := \bar{a}F'(r) \cup (R \cap F'(r))$, where $R \in L(A)$ fixed. Clearly F is weldefined, since F(r) changes only the indifference classes of F'(r). Therefore F has obviously all the social properties of F'.

Hence, F is a strongly non-dictatorial, strongly Pareto-optimal and positively associated welfare function from $L(A)_N$ to $L(A)$.

Suppose there is no $x,y \in A$, such that $\langle x,y\rangle$ is an inseparable pair of $L(A)_i$.

Then for all $x,y \in A$, with $L(A)_i \cap L(A,x>y) \neq \varphi$: there is a $z \in A$, and a $R' \in L(A)_i$ such that $xzy : R'$. 4.5.2.1

The proof succeeds in steps:

Step 1 If for some $R \in L(A)_i$ $xyz:R$, and $\langle x,y\rangle \in D(F,\{i\})$ or $\langle y,z\rangle \in D(F,\{i\})$, then $\langle x,z\rangle \in D(F,\{i\})$.

Suppose $\langle x,y\rangle \in D(F,\{i\})$ and $xyz : R$.

Take $r \in L(A)_N$ as follows: $xyz : R^i$, and

$yzx : R^j$, for $j \in N - \{i\}$.

By our assumption it follows that $xyz : F(r)$. Using the positive association of F it follows that $\langle x,z\rangle \in D(F,\{i\})$. The other case: $\langle y,z\rangle \in D(F,\{i\})$ follows similarly.

Step 2 If for some $R \in L(A)_i$, $xyz:R$, and $\langle x,z\rangle \notin D(F,\{i\})$, then $\langle x,y\rangle, \langle y,z\rangle \notin D(F,\{i\})$.

Follows immediate from step 1.

Step 3 If $\langle x,y\rangle \notin D(F,\{i\})$, then $\langle y,x\rangle \in D(F,N-\{i\})$. Is evident.

Step 4 If $\langle x,y\rangle, \langle y,z\rangle \in D(F,N-\{i\})$, then $\langle x,z\rangle \in D(F,N-\{i\})$.

Follows similarly as step 1.

Step 5 There is a pair $\langle x,y\rangle \in D(F,N-\{i\})$, such that $\langle y,x\rangle \in D(F,N-\{i\})$.

Since F is non-dictatorial there is a pair $\langle y_0,y\rangle \in A \times A$, such that $\langle y,y_0\rangle \notin D(F,\{i\})$ and $L(A)_i \cap L(A,y>y_0) \neq \varphi$. By step 3 this yields $\langle y_0,y\rangle \in D(F,N-\{i\})$.

Furthermore by (4.5.2.1) there is a $R_1 \in L(A)_i$, such that $y\ y_1\ y_0 : R_1$. Hence, by step 2 $\langle y,y_1\rangle, \langle y_1,y_0\rangle \notin D(F,\{i\})$ and by step 3 this implies $\langle y_1,y\rangle, \langle y_0,y_1\rangle \in D(F,N-\{i\})$.

Similarly there is a $y\ y_2\ y_1 : R_2 \in L(A)_i$ such that $\langle y_1,y_2\rangle, \langle y_2,y\rangle \in D(F,N-\{i\})$.

Succeeding this reasoning it follows that there are $\langle y_0,y_1\rangle, \langle y_1,y_2\rangle, \langle y_2,y_3\rangle, \ldots, \langle y_p,y\rangle \in D(F,N-\{i\})$, where $p = |A|$. Hence, there are $t_1 < t_2$, such that $y_{t_1} = y_{t_2}$.

Obviously, since $y\ y_{t_1+1}\ y_{t_1} : R_{t_1}$, $t_2 \geq t_1 + 2$.

Moreover, by step 4 it follows that $\langle y_{t_2},y_{t_1+1}\rangle \in D(F,N-\{i\})$

Take $\{x,y\} = \{y_{t_2},y_{t_1+1}\}$ and we are done.

Let $B_0 = \{x,y\}$, where x,y satisfies step 5.

Take $B_k = B_{k-1} \cup \{z \in A :$ there are $a,b \in \cup\{B_i : i < k\}$ and $R \in L(A)$, such that $a > z > b : R\}$.

Since A is finite there is a B, such that $B = B_k$ for all $k \geq p$, where $p = |A|$.

Step 6 By induction on k we will now prove that for all $x,y \in B$, with $x \neq y : \langle x,y\rangle \in D(F,N-\{i\})$.

Basis: trivial.

Induction step: suppose for all $x,y \in B_k, x \neq y$: $\langle x,y\rangle \in D(F,N-\{i\})$. Let $a,b \in B_{k+1}, a \neq b$. We have to prove $\langle a,b\rangle \in D(F,N-\{i\})$. Since $a,b \in B_{k+1}$, there are $a_0,a_1,b_0,b_1 \in \cup\{B_i : i \leq k\}$ and R_1, $R_2 \in L(A)_i$, such that $a_0>a>a_1 : R_1$ and $b_0>b>b_1 : R_2$. By the induction hypotheses it follows that $\langle a_1,a_0\rangle$, $\langle b_1,b_0\rangle$, $\langle a_0,a_1\rangle$, $\langle b_0,b_1\rangle \in D(F,N-\{i\})$. (4.5.2.2)

Obviously $\langle a_0,a_1\rangle$, $\langle b_0,b_1\rangle \notin D(F,\{i\})$. Hence, by step 2 $\langle a_1,a\rangle$, $\langle b,b_1\rangle \notin D(F,\{i\})$ and by step 3 $\langle a,a_1\rangle$, $\langle b_1,b\rangle \in D(F,N-\{i\})$. If $a_1 = b_1$, we are done by step 4. If

$a_1 \neq b_1$, we are done by the induction hypothesis and step 4. Hence, for all $x,y \in B$, $x \neq y$: $\langle x,y\rangle \in D(F,N-\{i\})$. By step 5 it follows that $|B| > 1$. By definition it follows that B is a separable set of $L(A)_1$. We are ready when $B \subset A$. If $B = A$, then by step 6 it follows that $\bar{n}(A \times A) \subseteq D(F,N-\{i\})$. But obviously by lemma 4.5.5, a lemma which is still to prove, then there is a non-dictatorial and Pareto-optimal welfare function from $L_{n-1}(A)$ to $L(A)$, where $|A| \geq 3$. This cannot be by our knowledge according to § 4.3. Hence, $B \subset A$ and we are done.

■

Before stating and proving lemma 4.5.5, used in the proof of theorem 4.5.2, we will make some remarks on this theorem. First of all it appears that by restricting only one set of individual orderings, we can create possibilities. On the other hand, this restriction should separate a set (not trivial) or preference between two alternatives from the rest. The names of inseparable pair and set intuitive clarify how these possibilities can occur. This separation allows disjoint coalitions to decide on the disjoint (separated) parts of the domain. Hereby non-dictatorship follows easily. The proof of theorem 4.5.2 is completed by

<u>Lemma 4.5.5</u>

Let $\Gamma = \langle A,N\rangle$ be a society and $S \subseteq N$, such that:

4.5.5.1 $L(A)_N$ is such that for all $i,j \in S$ there are $a,b \in A$, $R_1,R_2 \in L(A)_i$ and $R_3,R_4 \in L(A)_j$, with $a>b : R_1$, $b>a : R_2$, $a>b : R_3$ and $b>a : R_4$,

4.5.5.2 F is a strongly Pareto-optimal, strongly non-dictatorial, and positively associated welfare function from $L(A)_N$ to $L(A)$, and

4.5.5.3 $D(F,S) = K(L(A)_N,S)$.

Then there is a welfare function H on $\langle A,S\rangle$ from $L(A)_S$ to $L(A)$, which is strongly Pareto-optimal, strongly non-dictatorial and positively associated.

Proof of lemma 4.5.5

Without loss of generality suppose $S = \{1,2,\ldots,s\}$.

For $\tilde{r} \in L(A)_{N-S}$ and $r \in L(A)_S$ let $\langle r,\tilde{r}\rangle$ be the profile $\bar{r}$ in $L(A)_N$, such that $\bar{R}^i = R^i$, for all $i \in S$, and $\bar{R}^i = \tilde{R}^i$, for all $i \in N-S$.

Take $\tilde{r} \in L(A)_{N-S}$. Define $H_{\tilde{r}} : L(A)_S \to L(A)$ for all $r \in L(A)_S : H_{\tilde{r}}(r) := F(\langle r,\tilde{r}\rangle)$.

By (4.5.5.3) $H_{\tilde{r}}$ is strongly Pareto-optimal and positively associated.

Hence, we are done if there is a profile $\tilde{r} \in L(A)_{N-S}$ such that $H_{\tilde{r}}$ is strongly non-dictatorial.

Suppose for all $\tilde{r} \in L(A)_{N-S}$ there is an individual $i_{\tilde{r}}$ in S, which is a weak dictator of $H_{\tilde{r}}$.

Since $F(\langle r,\tilde{r}\rangle) \in L(A)$ it follows that for all $\tilde{r} \in L(A)_{N-S}$ and for all $r \in L(A)_S$: $F(\langle r,\tilde{r}\rangle) = R_{i_{\tilde{r}}}$.

It suffices to prove that, if $\tilde{r}$, $\bar{\bar{r}} \in L(A)_{N-S}$ differ in at most one coordinate, then $i_{\tilde{r}} = i_{\bar{\bar{r}}}$. Since then F is dictatorial which cannot be the case. Take $\tilde{r}$, $\bar{\bar{r}} \in L(A)_{N-S}$ such that they differ only in coordinate t, and suppose $i_{\tilde{r}} \neq i_{\bar{\bar{r}}}$.

We deduce a contradiction.

Without loss of generality suppose $i_{\tilde{r}} = 1$ and $i_{\bar{\bar{r}}} = 2$.

By assumption 4.5.5.1 there are $a,b \in A$, $R_1,R_2 \in L(A)_1$, $R_3,R_4 \in L(A)_2$, such that $a>b : R_1$, $b>a : R_2$, $a>b : R_3$ and $b>a : R_4$.

Take $\hat{r} \in L(A)_{S-\{1,2\}}$ and the following four profiles:

$r^1 := \langle R_1, R_4, \hat{r}, \tilde{r}\rangle$,
$r^2 := \langle R_1, R_4, \hat{r}, \bar{\bar{r}}\rangle$,
$r^3 := \langle R_2, R_3, \hat{r}, \tilde{r}\rangle$, and
$r^4 := \langle R_2, R_3, \hat{r}, \bar{\bar{r}}\rangle$.

Obviously $a>b : R_1 = F(r^1)$, $b>a : F(r^2) = R_4$
$b>a : R_2 = F(r^3)$ and $a>b : F(r^4) = R_3$.

Now there are four cases with respect to the preference between a and b in $\tilde{R}^t$ and $\bar{\bar{R}}^t$.

Case 1 $a>b : \tilde{R}^t$ and $a>b : \bar{\bar{R}}^t$

Then $r^1|_{\{a,b\}} = r^2|_{\{a,b\}}$, but $F(r^1)|_{\{a,b\}} \neq F(r^2)|_{\{a,b\}}$.

Hence, this cannot be the case since F is independent of irrelevant alternatives.

Case 2 $b>a : \tilde{R}^t$ and $b>a : \bar{\bar{R}}^t$ is similar to case 1.

Case 3 $a>b : \tilde{R}^t$ and $b>a : \bar{\bar{R}}^t$

Then the preference b above a is preserved when going from r^3 to r^4, but $b>a : F(r^3)$ and $a>b : F(r^4)$. Hence, this can either be the case, since it would imply that F is not positively associated.

Case 4 $b>a : \tilde{R}^t$ and $a>b : \bar{\bar{R}}^t$ is similar to case 3.

This completes the proof.

■

We will now discusse another class of domain restrictions. Let $\tilde{R}$ be a quasi-order on A, that is $\tilde{R}$ is $\langle\bar{a}^2,\bar{a}\rangle$-transitive and strongly complete. Let $L(A,\tilde{R}) := \{R \in L(A) : R \subseteq \tilde{R}\}$. Hence, $L(A,\tilde{R})$ is the set of linear orderings, which are contained in $\tilde{R}$. Equivalently, $L(A,\tilde{R})$ is the set of linear orderings that contain $\bar{a}\tilde{R}$. For every $R \in L(A,\tilde{R})$ we know that $\bar{a}\tilde{R} \subseteq R$. Hence, we could interpret $\tilde{R}$ as the a priori information, which we have about the set $L(A,\tilde{R})$.

Now suppose for every $i \in N$ there is a quasi-order $\tilde{R}_i \in Q(A)$ such that $L(A,\tilde{R}_i) = L(A)_i$, we can interpret $\langle\tilde{R}_1,\tilde{R}_2,\ldots,\tilde{R}_n\rangle$ as the a priori information, which is known about the set of profiles. In that case $L(A,\tilde{r})_N := L(A,\tilde{R}_1) \times L(A,\tilde{R}_2) \times\ldots L(A,\tilde{R}_n)$. Such domains are called <u>domains with a priori information</u>. See also Storcken [1986]). We know already that $L(A,x>y) = \{R \in L(A) : x>y : R\}$. $L(A,x>y) = L(A, A\times A - \{\langle y,x\rangle\})$, for $x \neq y$, is a set with a priori information. Furthermore, note that if $\tilde{R} = A \times A$, then $L(A,\tilde{R}) = L(A)$, hence we have no information, and if $\tilde{R} \in L(A)$, then $L(A,\tilde{R}) = \{\tilde{R}\}$ and we are completely informed. By these two examples it is evident that the information, known a priori, varies between knowing nothing and knowing everything. $\bar{a}\tilde{R}$ is also the only information, which we have on the relations in $L(A,\tilde{R})$, since it is the only invariant part of every relation in $L(A,\tilde{R})$.

First we deduce a characterizing property, which is used in other theorems on this subject of domains with a priori information.

Let $\Gamma = \langle A,N\rangle$ be a society, $\tilde{r} \in Q_n(A)$ and $L(A,\tilde{r})_N$ a domain with a priori information $\tilde{r}$. For all $x,y \in A$, $x \neq y$:

(1) $C(\tilde{r},xy) := \{i \in N : \langle x,y\rangle \in \bar{a}\tilde{R}^i\}$ is the set of individuals, who strictly prefer x to y in all profiles in $L(A,\tilde{r})_N$, and

(2) $L(\tilde{r},xy) := \{i \in N : \langle x,y\rangle \notin \bar{a}\tilde{R}^i \vee \langle y,x\rangle \notin \bar{a}\tilde{R}^i\} = N - (C(\tilde{r},xy) \vee C(\tilde{r},yx))$ are those individuals who liberately can prefer x to y or y to x.

Obviously $C(\tilde{r},xy)$, $C(\tilde{r},yx)$ and $L(\tilde{r},xy)$ is a partition of N and $L(\tilde{r},xy) = L(\tilde{r},yx)$. Moreover, for all $S \subseteq N$ we have that $K(L(A,\tilde{r})_N,S) = K(\tilde{r},S) := \{\langle x,y\rangle \in A\times A : C(\tilde{r},xy) \subseteq S \subseteq N-C(\tilde{r},yx)\}$. $K(\tilde{r},S)$ consists of those pairs $\langle x,y\rangle$, such that no constantly prefering y to x individual is in S and all constantly prefering x to y are in S.

<u>Theorem 4.5.6</u>

Let $\Gamma = \langle A,N\rangle$ be a society, $\tilde{r} \in Q_n(A)$ and $L(A,\tilde{r})_N$ a domain with a priori information. Adopt the following abbreviations: $C_{xy} := C(\tilde{r},xy)$, $K(S) := K(\tilde{r},S)$ and $L_{xy} := L(\tilde{r},xy)$.

Then (4.5.6.1) and (4.5.6.2) are equivalent.

4.5.6.1 There is a positively associated, strongly Pareto-optimal, and strongly non-dictatorial welfare function F on Γ from $L(A,\tilde{r})_N$ to $L(A)$.

4.5.6.2 There is a (decisiveness) function $D : 2^N \to 2^{\bar{n}(A\times A)}$ satisfying (4.5.6.2a) up to (4.5.6.2f).

4.5.6.2a For all $S \subseteq N : D(S) \subseteq K(S)$.

4.5.6.2b $D(N) = K(N)$.

4.5.6.2c For all $i \in N$ there is a $T \subseteq N-\{i\}$, such that $D(T) \neq \varphi$.

4.5.6.2d For all $S \subseteq N : D(S) = K(S) \cap \overline{cv}D(N-S)$.

4.5.6.2e For all $S \subseteq T \subseteq N : D(S) \cap K(T) \subseteq D(T)$.

4.5.6.2f For all $S,T \subseteq N$, all $\langle x,y\rangle \in D(S)$ and all $\langle y,z\rangle \in D(T)$: if $S \cap T \cap C_{zx} = \varphi$, then $\langle x,z\rangle \in D(C_{xz} \cup (S \cap T))$.

<u>Proof of theorem 4.5.6</u>

(4.5.6.1) → (4.5.6.2) Suppose (4.5.6.1).

Define $D(S) := D(F,S)$.

We will prove that D satisfies (4.5.6.1a) up to (4.5.6.1f).

(4.5.6.2a) follows by the definition of D.

(4.5.6.2b) follows from $K(L(A,\tilde{r})_N,N) = K(N)$ and F is Pareto-optimal.

(4.5.6.2c) Let $i \in N$. Then there is a profile $r \in L(A,\tilde{r})_N$, such that $x > y : F(r)$ and $y > x : R^i$ for some $x,y \in A$, since F is strongly non-dictatorial and $F(L(A,\tilde{r})_N) \subseteq L(A)$. Let

$T = \{j \in N : x > y : R^j\}$. Then $i \notin T$, $C_{xy} \subseteq T \subseteq N - C_{yx}$ and $\langle x,y\rangle \in D(F,T)$ by the possitive association of F. Hence, $\langle x,y\rangle \in D(T)$ and $T \subseteq N - \{i\}$.

(4.5.6.2d) Notice that, since $F(L(A,r)_N) \subseteq L(A)$, for all $S \subseteq N$ and $\langle x,y\rangle \in K(S)$:

$\langle x,y\rangle \in D(F,S) \leftrightarrow \langle y,x\rangle \notin D(F,N-S)$, and

$\langle x,y\rangle \in K(S) \leftrightarrow \langle y,x\rangle \in K(N-S)$, the proof is obvious.

(4.5.6.2e) follows immediately from the positively association.

(4.5.6.2f) Let $\langle x,y\rangle \in D(S)$, $\langle y,z\rangle \in D(F)$ and $C_{zx} \cap S \cap T = \varphi$.

Since $\langle x,y\rangle \in K(S)$ it follows $C_{xy} \subseteq S \subseteq N - C_{yx}$.

Similarly we have $C_{yz} \subseteq T \subseteq N - C_{zy}$.

Now take $\hat{r} \in L(A,r)$, such that:

$z>x>y : \hat{R}^i$, for all $i \in (S-T) \cap (N-C_{xz})$,

$x>z>y : \hat{R}^i$, for all $i \in (S-T) \cap C_{xz}$,

$y>z>x : \hat{R}^i$, for all $i \in (T-S) \cap (N-C_{xz})$,

$y>x>z : \hat{R}^i$, for all $i \in (T-S) \cap C_{xz}$,

$z>x : \hat{R}^i$, for all $i \in (N-(S\cup T)) \cap (N-C_{xz})$,

$x>z : \hat{R}^i$, for all $i \in (N-(S\cup T)) \cap C_{xz}$, and

$x>y>z : \hat{R}^i$, for all $i \in (S\cap T)$.

Note that $S \cap T \subseteq N - [C_{yx} \cup C_{zy}]$ and since $S \cap T \cap C_{zx} = \varphi$, such a profile exists.

Now by the positive association of F we have $x>y : F(\hat{r})$.

Hence, $x>z : F(\hat{r})$, since $F(\hat{r}) \in L(A)$.

But then by the positive association of F we have

$\langle x,z\rangle \in D(F, (S \cap T) \cup C_{xz})$.

Obviously $\langle x,z\rangle \in K(C_{xz} \cup (S \cap T))$.

Hence, by definition $\langle x,z\rangle \in D(C_{xz} \cup (S \cap T))$.

(4.5.6.2) $\rightarrow$ (2.5.6.1) Suppose (4.5.6.2)

Define for all $r \in L(A,r)_N$ and $x,y \in A$:

$\langle x,y\rangle \in F(r)$, iff either $x = y$,

or $x \neq y$ and $\langle x,y\rangle \in D(\{i \in N: x>y : R^i\})$.

<u>Claim 1</u> For all $r \in L(A,r)_N : F(r) \in L(A)$.

Let $r \in L(A,r)_N$. It is sufficient to prove that $F(r)$ is reflexive, antisymmetric, complete and transitive. $F(r)$ is reflexive by definition.

Let $\langle x,y\rangle \in \bar{n}(A\times A)$. By 4.5.6.2d it follows:

$\langle x,y\rangle \in D(S) \leftrightarrow \langle x,y\rangle \in K(S)$ & $\langle y,x\rangle \notin D(N-S)$

$\leftrightarrow \langle y,x\rangle \in K(N-S)$ & $\langle y,x\rangle \notin D(N-S)$.

Obviously $\langle x,y\rangle \in K(\{i \in N : x > y : R^i\})$.

Hence, $\langle y,x\rangle \notin F(r) \leftrightarrow \langle x,y\rangle \in F(r)$.

This proofs both antisymmetry and completeness of F.

Suppose $\langle x,y\rangle \in F(r)$, $\langle y,z\rangle \in F(r)$ and $|\{x,y,z\}| = 3$.

To prove the transitivity of F(r) it suffices to prove that $\langle x,z\rangle \in F(r)$.

By definition it follows: $\langle x,y\rangle \in D(\{i \in N: x > y : R^i\})$ and
$\langle y,z\rangle \in D(\{i \in N: y > z : R^i\})$.

Let $S := \{i \in N : x > y : R^i\}$ and $T := \{i \in N : y > z : R^i\}$.

Obviously $S \cap T \cap C_{zx} = \varphi$ and

$(S \cap T) \cup C_{xz} \subseteq \{i \in N : xz : R^i\} \subseteq (N - C_{zx})$, since L(A) is transitive.

Hence, $\langle x,z\rangle \in D(C_{xz} \cup (S \cap T))$ (by 4.5.6.2f) and by (4.5.6.2e) we are done.

<u>Claim 2</u> F is strongly Pareto-optimal by (4.5.6.2b).

<u>Claim 3</u> F is strongly non-dictatorial by (4.5.6.2c).

<u>Claim 4</u> F is positively associated. Take $r,\hat{r} \in L(A,\tilde{r})_N$ and $x,y \in A$, such that

$S := \{i \in N : x > y : R^i\} \subseteq \{i \in N : x > y : \hat{R}^i\} =: T$.

Suppose $x > y : F(r)$.

It suffices to prove that $x > y : F(\hat{r})$.

But this is obvious by (4.5.6.2e).

Claim 1 up to 4 prove (4.5.6.2) -> (4.5.6.1).

■

After this characterizing theorem our first aim is to prove that we can focus on social welfare functions F on societies with a priori information $\tilde{r}$, which satisfy the following condition $\bar{a}\tilde{R}^i \subseteq D(F,\{i\})$. For such situations it holds that C_{ab} is decisive on his a priori information.

<u>Theorem 4.5.7</u>

Suppose $\Gamma = \langle A,N\rangle$ is a society, $\tilde{r} \in Q_n(A)$, $L(A,\tilde{r})_N$ is a domain with a priori information $\tilde{r}$, $F : L(A,\tilde{r})_N \to L(A)$ is a strongly Pareto-optimal, strongly non-dictatorial, and positively associated welfare function on Γ.

Then there is a profile $\hat{r} \in Q_n(A)$, such that for all $i \in N$ $\tilde{R}^i \subseteq \hat{R}^i$ and there is a strongly Pareto-optimal, strongly non-dictatorial, and positively associated welfare function H from $L(A,\hat{r})_N$ to L(A) on Γ, such that $\bar{a}\hat{R}^i \subseteq D(H,\{i\})$.

Proof of theorem 4.5.7

Let F be such a welfare function and D as in (4.5.6.2).

Take $\hat{r} \in Q_n(A)$, such that for all $i \in N$ and $x,y \in A$:

$\langle x,y\rangle \in \hat{R}^i$, iff $\langle x,y\rangle \in \tilde{R}^i$ or $\langle y,x\rangle \notin D(C(\tilde{r},xy))$.

Define: $C'_{xy} := C(\tilde{r},xy)$, $C_{xy} := C(\hat{r},xy)$, $L'_{xy} := L(\tilde{r},xy)$,

$L_{xy} := L(\hat{r},xy)$, $K'(S) := K(\tilde{r},S)$, and $K(S) := K(\hat{r},S)$.

Define $D' : 2^N \to 2^{\bar{n}(A\times A)}$ for all $x,y \in A$ and all $S \subseteq N$:

$\langle x,y\rangle \in D'(S)$, iff $\langle x,y\rangle \in D((C_{xy} \cup S) - C_{yx})$ and

$\langle x,y\rangle \in K'(S)$.

Denote: $S_{xy} := (C_{xy} \cup S) - C_{yx}$ for all $S \subseteq N$.

It is sufficient to prove that D' satisfies (4.5.6.2).

(4.5.6.2a) is obvious by definition.

(4.5.6.2b) Let $\langle x,y\rangle \in K'(N)$. It satisfies to prove that $\langle x,y\rangle \in D'(N) = D((C_{xy} \cup N) - C_{yx}) = D(N - C_{yx})$. If $C'_{yx} = \varphi$, then $\langle x,y\rangle \in D(N - C_{yx})$ since $\langle y,x\rangle \notin D(C_{yx})$.

(4.5.6.2c) Let $i \in N$. Then there is a $T \subseteq N - \{i\}$ such that $\varphi \neq D(T)$. Hence, there is a $\langle x,y\rangle \in D(T) \cap K(T)$. Since then $T_{xy} = T$ it follows that $\langle x,y\rangle \in D(T_{xy}) = D'(T)$.

(4.5.6.2d) Let $\langle x,y\rangle \in \bar{n}(A\times A)$ and $S \subseteq N$. Then

$$\begin{aligned}
\langle x,y\rangle \in D'(S) &\Leftrightarrow \langle x,y\rangle \in D(S_{xy}) \;\&\; \langle x,y\rangle \in K'(S)\\
&\Leftrightarrow \langle x,y\rangle \in K(S_{xy}) \;\&\; \langle x,y\rangle \in K'(S)\\
&\qquad \&\; \langle x,y\rangle \in \overline{cv}D(N - S_{xy})\\
&\Leftrightarrow \langle x,y\rangle \in K'(S)\\
&\qquad \&\; \langle x,y\rangle \in \overline{cv}D((N - S)_{yx})\\
&\Leftrightarrow \langle x,y\rangle \in K'(S)\\
&\qquad \&\; \langle y,x\rangle \notin D((N - S)_{yx})\\
&\Leftrightarrow \langle x,y\rangle \in K'(S) \;\&\; \langle y,x\rangle \notin D(N-S)\\
&\Leftrightarrow \langle x,y\rangle \in K'(S) \cap \overline{cv}D'(N - S).
\end{aligned}$$

(4.5.6.2e) Let $\langle x,y\rangle \in D'(S)$, $S \subseteq T$, $\langle x,y\rangle \in K'(T)$.

It suffices to prove that $\langle x,y\rangle \in D'(T)$. Now $\langle x,y\rangle \in D(S_{xy})$, $S_{xy} \subseteq T_{xy}$, $\langle x,y\rangle \in K(T_{xy})$ and $\langle x,y\rangle \in K'(T)$. Hence, by (4.5.6.2e) we are done.

(4.5.6.2f) Suppose $\langle x,y\rangle \in D'(S)$, $\langle y,z\rangle \in D'(T)$, and $C'_{zx} \cap S \cap T = \varphi$.

We have to prove $\langle x,z\rangle \in D'(C'_{xz} \cup (S \cap T))$.

Since $C'_{zx} \cap S \cap T = \varphi$, it follows $\langle x,z\rangle \in K'(C'_{xz} \cup (S \cap T))$.

It suffices to prove $\langle x,z\rangle \in D([C'_{xz} \cup (S \cap T)]_{xz})$ 4.5.7.1

There are five cases.

<u>Case 1</u> $C'_{xz} \neq \varphi$ Then $C'_{xz} = C_{xz}$ and $\langle x,z\rangle \in D(C_{xz})$. By (4.5.6.2e) (4.5.7.1) follows evidently.

<u>Case 2</u> $C'_{zx} \neq \varphi$ Then $C'_{zx} = C_{zx}$ and $\langle z,x\rangle \in D(C_{zx})$.

Note that $C'_{zx} \cap S_{xy} \cap T_{xz} = C_{zx} \cap S_{xy} \cap T_{yz} = \varphi$.

Hence, $\langle x,z\rangle \in D(C_{xz} \cup (S_{xy} \cap T_{yz}))$ and $\langle z,x\rangle \notin D(C_{zx})$.

This contradicts an earlier result.

<u>Case 3</u> $C'_{xz} = C'_{zx} = \varphi$ and $C'_{xy} \neq \varphi$

Then $C'_{xy} = C_{xy}$ and $\langle x,y\rangle \in D(C_{xy})$ and $C_{xy} \subseteq S$.

Now $C_{xy} \cap T_{yz} \cap C_{zx} \subseteq C_{zy} \cap T_{yz} = \varphi$.

Hence, by (4.5.6.2f) it follows that $\langle x,z\rangle \in D(C_{xz} \cup (T_{yz} \cap C_{xy}))$.

$$\begin{aligned} \text{Now } C_{xz} \cup (T_{yz} \cap C_{xy})) &= C_{xz} \cup [\{(L_{yz} \cap T) \cup C_{yz}\} \cap C_{xy}] \\ &= C_{xz} \cup [C_{xy} \cap L_{yz} \cap T] \\ &= C_{xz} \cup [C_{xy} \cap L_{xz} \cap L_{yz} \cap T] \\ &\subseteq C_{xz} \cup [S \cap L_{xz} \cap T]. \end{aligned}$$

<u>Case 4</u> $C'_{xz} = C'_{zx} = \varphi$ and $C'_{yz} \neq \varphi$ is similar to case 3.

<u>Case 5</u> $C'_{xz} = C'_{zx} = \varphi$ and $C'_{yz} = C'_{xy} = \varphi$

Since $D(S_{xy}) \neq \varphi$ and $D(T_{xy}) \neq \varphi$ it follows that

$C'_{xz} = C'_{zx} = C'_{yz} = C'_{xy} = C'_{yx} = C'_{zy} = \varphi$.

By lemma 4.5.8 there is an individual $i \in N$ such that for all $\langle a,b\rangle \in \bar{n}(\{x,y,z\} \times \{x,y,z\})$ and all $M \subseteq N$: $\langle a,b\rangle \in D(M)$, iff $i \in M$. Clearly we are done.

■

In the proof of theorem 4.5.7 we used the following lemma.

<u>Lemma 4.5.8</u>

Suppose $\Gamma = \langle A,N\rangle$ is a society, $L(A,\tilde{r})$ a domain with a priori information $\tilde{r} \in Q_n(A)$, D a decisiveness function from 2^N to $2^{\bar{n}(A\times A)}$, which satisfies (4.5.6.2) and $B \subseteq A$, $|B| = 3$, such that for all $\langle x,y\rangle \in \bar{n}(B\times B)$: $\langle x,y\rangle \notin D(C_{xy})$.

Then there is an individual $\hat{\imath} \in N$, such that for all $S \subseteq N$ and $\langle x,y\rangle \in \bar{n}(B\times B) \cap K(S)$: $\langle x,y\rangle \in D(S) \leftrightarrow \hat{\imath} \in S$.

Proof of lemma 4.5.8

X_1 up to X_{27} is a partition of N, where

$X_1 := C_{xy} \cap C_{yz} \cap C_{xz}$, $X_{15} := L_{xy} \cap L_{yz} \cap C_{zx}$,
$X_2 := C_{xy} \cap C_{yz} \cap L_{xz} = \varphi$, $X_{16} := L_{xy} \cap C_{zy} \cap C_{xz} = \varphi$,
$X_3 := C_{xy} \cap C_{yz} \cap C_{zx} = \varphi$, $X_{17} := L_{xy} \cap C_{zy} \cap L_{xz}$,
$X_4 := C_{xy} \cap L_{yz} \cap C_{xz}$, $X_{18} := L_{xy} \cap C_{zy} \cap C_{zx}$,
$X_5 := C_{xy} \cap L_{yz} \cap L_{xz}$, $X_{19} := C_{yx} \cap C_{yz} \cap C_{xz}$,
$X_6 := C_{xy} \cap L_{yz} \cap C_{zx} = \varphi$, $X_{20} := C_{yx} \cap C_{yz} \cap L_{xz}$,
$X_7 := C_{xy} \cap C_{zy} \cap C_{xz}$, $X_{21} := C_{yx} \cap C_{yz} \cap C_{zx}$,
$X_8 := C_{xy} \cap C_{zy} \cap L_{xz}$, $X_{22} := C_{yx} \cap L_{yz} \cap C_{xz} = \varphi$,
$X_9 := C_{xy} \cap C_{zy} \cap C_{zx}$, $X_{23} := C_{yx} \cap L_{yz} \cap L_{xz}$,
$X_{10} := L_{xy} \cap C_{yz} \cap C_{xz}$, $X_{24} := C_{yx} \cap L_{yz} \cap C_{zx}$,
$X_{11} := L_{xy} \cap C_{yz} \cap L_{xz}$, $X_{25} := C_{yx} \cap C_{zy} \cap C_{xz} = \varphi$,
$X_{12} := L_{xy} \cap C_{yz} \cap C_{zx} = \varphi$, $X_{26} := C_{yx} \cap C_{zy} \cap L_{xz} = \varphi$,
$X_{13} := L_{xy} \cap L_{yz} \cap C_{xz}$, $X_{27} := C_{yx} \cap C_{zy} \cap C_{zx}$,
$X_{14} := L_{xy} \cap L_{yz} \cap L_{xz}$, and $B = \{x,y,z\}$.

Let now $S_1 := C_{xy} \cup X_{10} \cup X_{13} \cup X^1_{14} \cup X_{17}$,

$S_2 := C_{yz} \cup X_4 \cup X_{13} \cup X^2_{14} \cup X_{23}$, and

$S_3 := C_{zx} \cup X_8 \cup X^3_{14} \cup X_{17} \cup X_{23}$.

Where X^1_{14}, X^2_{14}, X^3_{14} are subsets of X_{14}.

Note that $\langle x,y\rangle \in K(S_1)$, $\langle y,z\rangle \in K(S_2)$ and $\langle z,x\rangle \in K(S_3)$.

By (4.5.6.2d) for each choice of X^1_{14}, X^2_{14}, X^3_{14} precisely one of the following 8 cases holds:

(α1) $\langle x,y\rangle \in D(S_1)$, $\langle y,z\rangle \in D(S_2)$ and $\langle z,x\rangle \in D(S_3)$, or
(α2) $\langle x,y\rangle \in D(S_1)$, $\langle y,z\rangle \in D(S_2)$ and $\langle x,z\rangle \in D(N-S_3)$, or
(α3) $\langle x,y\rangle \in D(S_1)$, $\langle z,y\rangle \in D(N-S_2)$ and $\langle z,x\rangle \in D(S_3)$, or
(α4) $\langle x,y\rangle \in D(S_1)$, $\langle z,y\rangle \in D(N-S_2)$ and $\langle x,z\rangle \in D(N-S_3)$, or
(α5) $\langle y,x\rangle \in D(N-S_1)$, $\langle y,z\rangle \in D(S_2)$ and $\langle z,x\rangle \in D(S_3)$, or
(α6) $\langle y,x\rangle \in D(N-S_1)$, $\langle y,z\rangle \in D(S_2)$ and $\langle x,z\rangle \in D(N-S_3)$, or
(α7) $\langle y,x\rangle \in D(N-S_1)$, $\langle z,y\rangle \in D(N-S_2)$ and $\langle z,x\rangle \in D(S_3)$, or
(α8) $\langle y,x\rangle \in D(N-S_1)$, $\langle z,y\rangle \in D(N-S_2)$ and $\langle x,z\rangle \in D(N-S_3)$.

Note that:

$S_1 \cap S_2 \subseteq C_{xz} \cup (X^1_{14} \cap X^2_{14})$,

$S_1 \cap S_3 \subseteq C_{zy} \cup (X^1_{14} \cap X^3_{14})$,

$S_2 \cap S_3 \subseteq C_{yz} \cup (X^2_{14} \cap X^3_{14})$,

$(N-S_1) \cap (N-S_2) \subseteq C_{zx} \cup (X_{14} - X^1_{14}) \cap (X_{14} - X^2_{14}))$,

$(N-S_1) \cap (N-S_3) \subseteq C_{yz} \cup (X_{14} - X^1_{14}) \cap (X_{14} - X^3_{14}))$, and

$(N-S_2) \cap (N-S_3) \subseteq C_{xy} \cup (X_{14} - X^2_{14}) \cap (X_{14} - X^3_{14}))$. 4.5.8.1.

Claim 1 There are $a,b \in B$, $a \neq b$, such that $\langle a,b\rangle \in D(C_{ab} \cup S)$ for some $S \subseteq X_{14}$.

Proof of claim 1 Take $X^1_{14} = X^2_{14} = X_{14} - X^3_{14}$.

Then (α1) and (α3) cannot be the case, since otherwise by (4.5.8.1 and 4.5.6.2f) $\langle z,y\rangle \in D(C_{zy} \cup (S_1 \cap S_3)) = D(C_{zy})$, which contradicts our assumptions.

Then (α1) and (α5) cannot be the case, since similarly it would follow that $\langle y,x\rangle \in D(C_{yx} \cup (S_2 \cap S_3)) = D(C_{yx})$, which is again contradicting our assumptions.

Then (α6) and (α8) cannot be the case, since this would lead to the contradicting result

$\langle y,z\rangle \in D(C_{yz} \cup ((N-S_1) \cap (N-S_3))) = D(C_{yz})$.

Then (α4) and (α8) cannot be the case, since this would yield contradicting result

$\langle x,y\rangle \in D(C_{xy} \cup ((N-S_2) \cap (N-S_3))) = D(C_{xy})$.

Hence, either (α2) or (α7) holds.

From (α2) it follows by (4.5.8.1) and (4.5.6.2f) that

$\langle x,z\rangle \in D(C_{xz} \cup (S_1 \cap S_2)) = D(C_{xz} \cup X^1_{14})$.

From (α7) it follows similarly that

$\langle z,x\rangle \in D(C_{zx} \cup ((N-S_1) \cap (N-S_2))) = D(C_{zx} \cup X^3_{14})$.

Which proves our claim.

Claim 2 If $T \subseteq X_{14}$ and $\langle a,b\rangle \in D(C_{ab} \cup T)$, then $\langle a,c\rangle \in D(C_{ac} \cup T)$ and $\langle c,b\rangle \in D(C_{cb} \cup T)$.

Proof of claim 2 Without loss of generality suppose $\langle x,y\rangle = \langle a,b\rangle$. Take $T = X^1_{14} = X^3_{14} = X_{14} - X^2_{14}$.

Then similarly as in the proof of claim 1 it follows either (α3) or (α6) is the case.

Obviously (α6) does not hold.

Hence, (α3) is the case and consequently $\langle z,y\rangle \in D(C_{zy} \cup T)$.

Similarly it follows that $\langle x,z\rangle \in D(C_{xz} \cup T)$, which proves this claim.

Claim 3 If $T \subseteq X_{14}$ and $\langle a,b\rangle \in D(C_{ab} \cup T)$, then for all $\langle p,q\rangle \in \bar{n}(B \times B) : \langle p,q\rangle \in D(C_{pq} \cup T)$.

Proof of claim 3 is classical in social choice theory and therefore regarded as obvious.

We have just proven the 'first' steps in the classical way of proven Arrow's impossibility theorem, but now on special restricted domains. We continue in the same classical way.

From the above it follows, that there is a $T \subseteq X_{14}$ such that for all $\langle a,b\rangle \in \bar{n}(B \times B)$ $\langle a,b\rangle \in D(C_{ab} \cup T)$ and is minimal with this property.

If $|T| = 1$ then we are done by (4.5.6.2d) and (4.5.6.2e).

Suppose $|T| \geq 2$. We will deduce a contradiction and are done.

Take $i \in T$.

Then neither $\langle a,b\rangle \in D(C_{ab} \cup \{i\})$ nor $\langle a,b\rangle \in D(C_{ab} \cup X_{14} - (T - \{i\}))$, for every $\langle a,b\rangle \in \bar{n}(B\times B)$.

Hence, $\langle x,y\rangle \in D(C_{xy} \cup (N - (X_{14} - (T - \{i\}))))$ and $\langle y,z\rangle \in D(C_{yz} \cup T)$.

Now $T \cap (N - (X_{14} - (T - \{i\}))) \cap C_{zx} = \varphi$.

Hence, $\langle x,z\rangle \in D(C_{xz} \cup (T \cap (N - (X_{14} - (T - \{i\}))))) = D(C_{xz} \cup (T - \{i\}))$, which contradicts the minimality of T.

■

We will now classify a special type of a priori restricted domains, which admit a strongly Pareto-optimal, strongly non-dictatorial, positively associated and symmetric decisive welfare function.

First we have to define symmetric decisiveness.

Definition 4.5.9

A welfare function F on a society $\underset{\sim}{\Gamma} \underset{\sim}{=} \langle A,N\rangle$ from the domain $L(A,r)_N$, with a priori information $r \in Q_n(A)$ is symmetric decisive, iff for all $\langle x,y\rangle \in A \times A$ and $S \subseteq N$:

if $\langle x,y\rangle \in D(F,S)$ and $\langle x,y\rangle \notin D(F,C_{xy})$, then $\langle y,x\rangle \in D(F,(S \cup C_{yx}) - C_{xy})$.

■

Obviously symmetric decisiveness coincides more or less with the symmetry property of §4.3. Because of the restricted domain approach it has such a strange appearance. A welfare function is symmetric decisive, iff a coalition S decides about $\langle x,y\rangle$ and $\langle x,y\rangle \notin D(F,C_{xy})$, then $S - C_{xy}$ is decisive about $\langle y,x\rangle$.

We have now the following characterizing theorem of domains with a priori information admitting symmetric decisive, strongly Pareto-optimal, strongly non-dictatorial, and positively associated welfare functions.

<u>Theorem 4.5.10</u>

Let $\Gamma = \langle A,N\rangle$ be a society and let $L(A,\tilde{r})_N$ be a domain with a priori information $\tilde{r} \in Q_n(A)$.

Then (4.5.10.1), (4.5.10.2) and (4.5.10.3) are equivalent.

4.5.10.1 There is a symmetric decisive, strongly Pareto-optimal, strongly non-dictatorial, and positively associated welfare function F on Γ from $L(A,\tilde{r})_N$ to $L(A)$.

4.5.10.2 There is a $\hat{r} \in Q_n(A)$ such that for all $i \in N$ $\tilde{R}^i \subseteq \hat{R}^i$, and a symmetric decisive, strongly Pareto-optimal, strongly non-dictatorial, and positively associated welfare function F on Γ from $L(A,\hat{r})$ to $L(A)$, such that for all $i \in N$ $\bar{a}\hat{R}^i \subseteq D(F,\{i\})$.

4.5.10.3 There is a relation $\bar{R} \in Q(A)$, such that for all $i \in N$: $\bar{a}\bar{R} \subseteq \bar{a}\tilde{R}$. Furthermore, there is a coalition labeling

$l : \bar{n}\bar{R} \rightarrow (2^{2^N - \{\varphi\}}) - \{\varphi\})$, satisfying (4.5.10.3a) up to (4.5.10.3g).

4.5.10.3a <u>The labeling is simple</u>, that is for all $\langle a,b\rangle \in \bar{n}\bar{R}$ and all $S \subseteq T \subseteq N$: $l(\langle a,b\rangle) \neq \varphi$ and if $S \in l(\langle a,b\rangle)$, then $T \in l(\langle a,b\rangle)$.

4.5.10.3b <u>The labeling is non-dictatorial</u>, that is for all $i \in N$, there is a pair $\langle a,b\rangle \in \bar{n}\bar{R}$ and a coalition $S \in l(\langle a,b\rangle)$, such that $i \notin S$.

4.5.10.3c <u>The labeling is pure</u>, that is for all $\langle a,b\rangle \in \bar{n}\bar{R}$ and all $S \in l(\langle a,b\rangle)$: $N - S \notin l(\langle a,b\rangle)$.

4.5.10.3d <u>The labeling is strong</u>, that is for all $\langle a,b\rangle \in \bar{n}\bar{R}$ and all $S \notin l(\langle a,b\rangle)$: $N - S \in l(\langle a,b\rangle)$.

4.5.10.3e <u>The labeling is constant on connected symmetric parts of</u> $\bar{n}\bar{R}$, that is for all $\langle a,b\rangle,\langle b,c\rangle \in \overline{ns\bar{R}}$: $l(\langle a,b\rangle) = l(\langle b,c\rangle)$.

4.5.10.3f <u>The labeling reflects the a priori information</u>, that is for all $\langle a,b\rangle \varepsilon \bar{a}\bar{R}$: $C(r, ab) \varepsilon l(\langle a,b\rangle)$.

4.5.10.3g <u>The labeling is transitive</u>, that is for all $\langle a,b\rangle,\langle b,c\rangle,\langle a,c\rangle \varepsilon \bar{R}$: if $S \varepsilon l(\langle a,b\rangle)$ and $T \varepsilon l(\langle a,b\rangle)$, then $S \cap T \varepsilon l(\langle a,c\rangle)$.

<u>Proof of theorem 4.5.10</u>

(4.5.10.1) -> (4.5.10.2).

Note that such a F exists, iff (4.5.6.2) holds and D satisfies the property: if $\langle x,y\rangle \varepsilon D(S)$ and $\langle x,y\rangle \not\varepsilon D(C_{xy})$, then $\langle y,x\rangle \varepsilon D(S_{yx})$.

Now this property is invariant under the construction of (4.5.7).

Hence, (4.5.10.2) follows evidently.

(4.5.10.2) -> (4.5.10.3).

Take $\bar{R} = \{\langle x,y\rangle \varepsilon A \times A : \langle x,y\rangle \varepsilon \hat{R}^i$ for all $i \varepsilon N\}$.

Evidently we have $\bar{a}\bar{R} \subseteq \bar{a}\tilde{R}^i$, for all $i \varepsilon N$.

Let $l : \bar{n}\bar{R} \rightarrow 2^{2^N}$ be defined as follows:

$l(\langle x,y\rangle) := \{S : \langle x,y\rangle \varepsilon D(S)\}$, where D is the decisiveness function according to (4.5.6.2).

Note that, if $\langle x,y\rangle \varepsilon \overline{ns\bar{R}}$ then $\langle x,y\rangle \varepsilon D(S) \leftrightarrow \langle y,x\rangle \varepsilon D(S)$ because of the symmetric decisiveness of F, and 4.5.10.4

if $\langle x,y\rangle \varepsilon \bar{a}\bar{R}$, then $\langle x,y\rangle \varepsilon D(C_{xy})$, since $\bar{a}\hat{R}^i \subseteq D(F,\{i\})$ and $\cup\{\bar{a}\hat{R}^i : i \varepsilon N\} = \bar{a}\bar{R}$. 4.5.10.5

l is simple because of (4.5.6.2e).

l is non-dictatorial because of (4.5.6.2c).

l is pure and strong because of (4.5.6.2d).

l is transitive because of (4.5.6.2f).

l reflects the a priori information because of (4.5.10.5).

Remains to prove that l is constant on connected parts in $\overline{ns\bar{R}}$.

Let $\langle a,b\rangle,\langle b,c\rangle \varepsilon \overline{ns\bar{R}}$.

Let $S \varepsilon l(\langle a,b\rangle)$ it is sufficient to prove that $S \varepsilon l(\langle b,c\rangle)$. There are four cases.

<u>Case 1</u> c = a We are done by (4.5.10.4).

<u>Case 2</u> $a \neq c$ and $\langle a,c\rangle \varepsilon \bar{a}\bar{R}$.

$$\begin{tikzcd} a \arrow[r, dash, "S"] \arrow[rr, bend right, "C_{ac}"'] & b \arrow[r, dash] & c \end{tikzcd}$$

Note that $S \in l(\langle b,a\rangle)$ and $C_{ac} \in l(\langle a,c\rangle)$.

Hence, by the transitivity of l it follows that $S \cap C_{ac} \in l(\langle b,c\rangle)$.

Hence, $S \in l(\langle b,c\rangle)$, because l is simple.

Case 3 $a \neq c$ and $\langle c,a\rangle \in \bar{a}\bar{R}$ is similar to case 2.

Case 4 $a \neq c$ and $\langle a,c\rangle \in \overline{ns\bar{R}}$.

There holds $T \in l(\langle a,c\rangle)$, for some $T \in 2^N$.

Hence, $T \cap S \in l(\langle b,c\rangle)$ and $S \in l(\langle b,c\rangle)$.

This completes the implication.

(4.5.10.3) -> (4.5.10.2) is straightforwardly to prove similar to the proof of (4.5.6.2) -> (4.5.6.1).

■

Theorem 4.5.10 characterizes those domains with a priori information, which admit symmetric decisive, Pareto-optimal, non-dictatorial and positively associated welfare functions. This is the case, iff it can assign to every pair in a quasi-order $\bar{R}$, contained in all the individual a priori information, a simple strong and pure game. Furthermore, this assignment should be constant on connected symmetrical parts of $\bar{R}$, transitive, non-dictatorial and a priori information respecting. Because of all these strong properties of the labelling it is rather easy to check whether or not such a `nice' welfare function exists.

Although we have many other results on this type of domain restriction the discussion is ended here since the deduction of these results is rather thechnical and very long. As a last remark it is pointed out that the results in Storcken [1986] can also be deduced by virtue of (4.5.6), which reduces the proofs there remarkably.

In the final part of this section domain restrictions will be discussed, that resemble investigated domain restrictions in literature. In Kalai & Muller [1977], Maskin [1976], Ritz [1984] and Ritz [1985] domain restrictions are discussed, where all the individuals have the same set of possible individual orderings. Hence, e.g., $L(A)_i = L(A)_j$ for all $i,j \in N$. Here we will discusse the case where the equality may be changed in an inclusion. So cases where $L(A)_1 \subseteq L(A)_2 \subseteq \dots \subseteq L(A)_n$ are discussed here. On the other hand, while making the structure of the domain more complicated we simplify the welfare function by imposing the positive association on it instead of the independence of irrelevant altenatives.

In Ritz [1985] (e.g.) the domain for a society, with two individual precisely, is charaterized in order to admit a strongly Pareto-optimal and strongly non-dictatorial welfare function. In Kalai & Muller [1977] and Ritz [1983] it is shown for societies, with n individuals, and domains in which the sets of possible individual orderings are equal, admit such welfare function, iff in that domain there are two individuals whose 'subdomain' (i.e. the domain and the society restricted to these two individuals) admit such welfare functions. Hereby the general case for this type of domains is characterized.

In principal we will show the same. In lemma 4.5.5 it is shown that if a subgroup $S \subseteq N$ is completely decisive on all of its feasible unanimous preferences, then the welfare function restricted to S is 'nice', whenever that on N is 'nice'. The technique shown there is to 'freeze' all preferences in N-S, like it is done by e.g. Ritz [1983]. This is possible because of the decisiveness of S by this `freezening' the obtained welfare function remains strongly Pareto-optimal. Here, however, we will use another thechnique, introduced by Kalai & Muller [1977]. They try to decrease the number of individuals admitting 'nice' welfare functions on their domain, by forcing to act two individuals as one. The prove of Kalai & Muller [1977] is different from that exposed, here because the difference in the domain. In their paper the set of individual orderings is in L(U). Here it is in W(U). That is also the reason, why the stronger positive association is chosen here instead of the independence property.

We will first prove that for two individuals the independence condition is equivalent to the positive association.

Theorem 4.5.11

Let $\Gamma = \langle A,N \rangle$ be a society, with $N = \{1,2\}$. Furthermore, let V and W be classified sets of strongly complete orderings, and let F be a strongly Pareto-optimal welfare function on Γ from $V(A)_N$ to $W(A)$. Then (4.5.11.1) and (4.5.11.2) are equivalent.

4.5.11.1 F is independent of irrelevant alternatives.

4.5.11.2 F is positively associated.

Proof of theorem 4.5.11

(4.5.11.2) $\rightarrow$ (4.5.11.1) The positive association is just stronger than the independence condition (See (1.4.7)).

(4.5.11.1) $\rightarrow$ (4.5.11.2) Let $x,y \in A$ $\bar{r},\tilde{r} \in V(A)_N$ such that for all $i \in N$:

if $\langle x,y \rangle \in \bar{R}^i$, then $\langle x,y \rangle \in \tilde{R}^i$ and

if $\langle x,y \rangle \in \bar{a}\bar{R}^i$, then $\langle x,y \rangle \in \bar{a}\tilde{R}^i$.

It is sufficient to prove that

if $x > y : F(\bar{r})$, then $x > y : F(\tilde{r})$.

Suppose $x > y : F(\bar{r})$.

There are two cases.

Case 1 $x \geq y : \bar{R}^i$ for all $i \in N$ and $x > y : \bar{R}^j$ for some $j \in N$. Then by our assumptions: $x \geq y : \tilde{R}^i$ and $x > y : \tilde{R}^j$. Hence, we are done by the strong Pareto-optimality.

Case 2 $\{i,j\} = \{1,2\}$, $x \geq y : \bar{R}^i$ and $y > x : \bar{R}^j$. Then $x > y : \tilde{R}^i$. If $x > y : \tilde{R}^j$, then we are done by the Pareto-optimality. If $y > x : \tilde{R}^j$, then we are done by the independence of irrelevant alternatives.

Case 3 $x \sim y : \bar{R}^i$ for all $i \in N$. Then $x \geq y : \tilde{R}^i$ for all $i \in N$. If $x > y : \tilde{R}^j$ for some $j \in N$ we are done by the Pareto-optimality. If $x \sim y : \tilde{R}^j$ for all $j \in N$, we are done with the independence of irrelevant alternatives.

■

The next theorem states that a welfare function with range W(A) can be transformed in one to with range L(A) leaving various properties invariant.

Theorem 4.5.12

Let $\Gamma = \langle A,N \rangle$ be a society and $V(A)_i \subseteq V(A)$, for all $i \in N$, where V is classified as a set of complete orderings.

Then (4.5.12.1) and (4.5.12.2) are equivalent.

4.5.12.1 There is a strongly Pareto-optimal, strongly non-dictatorial and positively associated welfare function F on Γ from $V(A)_N$ to $W(A)$.

4.5.12.2 There is a strongly Pareto-optimal, strongly non-dictatorial and positively associated welfare function H on Γ from $V(A)_N$ to $L(A)$.

Proof of theorem 4.5.12

(4.5.12.2) -> (4.5.12.1) Evident since $L(A) \subseteq W(A)$.

(4.5.12.1) -> (4.5.12.2) Let F as in (4.5.12.1). Take $\hat{R} \in L(A)$.

Define $H(r) := \bar{a}F(r) \cup (\bar{s}F(r) \cap \hat{R})$.

H changes only symmetric parts of F on a constant way.

It is therefore strongly Pareto-optimal, strongly non-dictatorial, and positively associated.

■

Now we will prove that welfare functions on a domain with n-1 individuals can be extended to a domain with n individuals, such that several social conditions are invariant.

Theorem 4.5.13

Suppose $\Gamma = \langle A,N \rangle$ is a society, with $n = |N| > 2$, $\Gamma' = \langle A,N-\{n\} \rangle$, $V(A)_i \subseteq V(A)$ for all $i \in N$, V is a classified set of complete orderings and there is a welfare function F from $V(A)_{N-\{n\}}$ to $L(A)$ on Γ', which is simultaniously strongly Pareto-optimal, strongly non-dictatorial and positively associated.

Then there is a welfare function H from $V(A)_N$ to $L(A)$ on Γ, which is strongly Pareto-optimal, strongly non-dictatorial and positively associated.

Proof of theorem 4.5.13

Suppose F as above. By (4.5.12) suppose without loss of generality that $F : V(A)_{N-\{n\}} \to L(A)$.

Take $H : V(A)_N \to L(A)$, with $\langle R^1,R^2..R^n \rangle \to F(\langle R^1,..R^{n-1} \rangle)$.

Clearly H is strongly Pareto-optimal, strongly non-dictatorial and possitively associated, since n is just a dummy addition to F.

■

Because of this dummy addition H is not so appreciable, although in very large societies one might think that the addition of 1 individual has no effect.

We will now consider the reverse of theorem 4.15.3. That is having a `nice' H does there exist a `nice' F, where F and H are as in (4.5.13)? The general case of this question is difficult to answer. Therefore, we concentrate on a special type of domain restrictions. (See theorem 4.5.14).

As a notational convention let $\{R_{A|_B} : R_A \in V(A)_i\} =: V(B)_i$ for all $\varphi \neq B \subseteq A$.

Theorem 4.5.14

Suppose $\Gamma = \langle A,N \rangle$ is a society, with $|N| = n \geq 4$, V is a classified set of complete orderings, $V(A)_N$ is a restricted domain, and i_1, i_2, j_1, j_2 are four different individuals in N. Suppose furthermore that

4.5.14.1 for all $B \subseteq A$, with $|B| = 3$: $V(B)_{i_1} \subseteq V(B)_{j_1}$ and $V(B)_{i_2} \subseteq V(B)_{j_2}$, and

4.5.14.2 for all $t_1, t_2 \in N$: There are $x,y \in A$, such that $x \neq y$: $L(\{x,y\}) \subseteq V(\{x,y\})_{t_1}$ and $L(\{x,y\}) \subseteq V(\{x,y\})_{t_2}$.

Then (4.5.14.3) and (4.5.14.4) are equivalent.

4.5.14.3 There is a strongly Pareto-optimal, strongly non-dictatorial and positively associated welfare function H on Γ from $V(A)_N$ to W(A).

4.5.14.4 There is a strongly Pareto-optimal, strongly non-dictatorial and positively associated welfare function F on $\Gamma^i = \langle A,N_i \rangle$ from $V(A)_{N_i}$ to W(A), for some $i \in N$, where $N_i := N - \{i\}$.

Proof of theorem 4.5.14

(4.5.14.4) -> (4.5.14.3) is trivial by theorem 4.5.13 and 4.5.12.

(4.5.14.3) -> (4.5.14.4)

We will prove that F exists for $i = j_1$ or $i = j_2$.

We will first define F^{j_k}, such that j_k and i_k play the same rôle in H for $k = \{1,2\}$, and then prove that one of them is non-dictatorial.

To avoid unnecessary variables suppose without loss of generality that $i_1 = 1$, $j_1 = 2$, $i_2 = 3$ and $j_2 = 4$.

Furthermore, by (4.5.12) suppose without loss of generality that $H : V(A)_N \to L(A)$. Define F^2 for all $\bar{r} \in V(A)_{N_2}$ and all

$x,y \in A : \langle x,y\rangle \in F^2(\bar{r})$, iff there is a profile $\hat{r} \in V(A)_N$, with for all $i \geq 3$, $\hat{R}^i|_{\{x,y\}} = \bar{R}^i|_{\{x,y\}}$ and $\hat{R}^1|_{\{x,y\}} = \hat{R}^2|_{\{x,y\}} = \bar{R}^1|_{\{x,y\}}$ and $\langle x,y\rangle \in H(\hat{r})$.

Since H is independent of irrelevant alternatives F^2 is well-defined. Furthermore, it follows evidently that $F^2(\bar{r})$ is reflexive, antisymmetric and complete, since $H(\hat{r})$ is so for all $\hat{r} \in V(A)_N$.

Claim 1 $F^2 : V(A)_{N_2} \to L(A)$.

It suffices to prove that $F^2(\bar{r})$ is transitive for all $\bar{r} \in L(A)_{N_2}$. Take $\bar{r} \in L(A)_{N_2}$. Let $a > b : F^2(\bar{r})$ and $b > c : F^2(\bar{r})$. It suffices to prove that $a > c : F^2(\bar{r})$.

Take $\hat{r} \in L(A)_N$, such that $\bar{R}^i = \hat{R}^i$ for all $i \in N - \{2\}$ and $\hat{R}^2|_{\{a,b,c\}} = \bar{R}^1|_{\{a,b,c\}}$. Because of (4.5.14.1) such a profile exists. Using the definition of F^2 and H it follows that: $a > c : F^2(\bar{r})$.

Claim 2 F^2 is strongly Pareto-optimal and positively associated.

This follows from the assumptions of H and the definition of F^2 in an evident way.

Similarly F^4 can be defined. F^4 has the same properties as F^2.

We will now prove that not both F^2 and F^4 can be dictatorial. Let t be a weak dictator of F^2. Since

$F^2(V(A)_{N_2}) \subseteq L(A)$ t is a strong dictator of F^2. Obviously since H is non-dictatorial using (4.5.14.2) it follows that $t = 1$.

Similarly only 3 can be strong dictator of F^4.

Again using (4.5.14.2) it cannot be the case that 1 and 3 are both dictators of F^2 and F^4 respectively.

Hence, F^2 or F^4 satisfies (4.5.14.4).

■

As an immediate result we have the following theorem:

Theorem 4.5.16

Suppose $\Gamma = \langle A,N \rangle$ is a society, with $|N| = n \geq 3$, V is a classified set of complete orderings, $V(A)_N$ is a restricted domain, such that

4.5.16.1 for all $B \subseteq A$, with $|B| = 3$:
$V(B)_1 \subseteq V(B)_2 \subseteq V(B)_3 \subseteq \ldots \subseteq V(B)_n$, and

4.5.16.2 there are $x,y \in A$, $x \neq y$, with $L(\{x,y\}) \subseteq V(\{x,y\})_1$.

Then (4.5.16.3) and (4.5.16.4) are equivalent.

4.5.16.3 There is a strongly Pareto-optimal, strongly non-dictatorial, and positively associated welfare function H on Γ from $V(A)_N$ to $W(A)$.

4.5.16.4 There is a subset $M \subseteq N$, such that $|M| = 3$, and a strongly Pareto-optimal, strongly non-dictatorial, and positively associated welfare function F on Γ' from $V(A)_M$ to $W(A)$, where $\Gamma' = \langle A,M \rangle$.

Proof of theorem 4.5.16

(4.5.16.4) -> (4.5.16.3) follows from (4.5.13) and (4.5.12).

(4.5.16.3) -> (4.5.16.4) follows from repeatedly application of (4.5.14).

■

In general in theorem 4.5.16 we can choose $W(A)_N$ for $V(A)_N$ and allow the individual orderings to have symmetric parts, i.e., indifference classes. On the otherhand, if this condition is dropped, so $V(A) = L(A)$, by virtue of (4.5.5), stronger results become deducable.

Theorem 4.5.18

Suppose $\Gamma = \langle A,N \rangle$ is a society, with $|N| \geq 2$ and $\hat{\imath},\hat{\jmath} \in N$, and $L(A)_N$ is a restricted domain, such that

4.5.18.1 for all $k \in N$:

for all $B \subseteq A$, with $|B| = 3$, $L(B)_{\hat{\imath}} \subseteq L(B)_k$, or

for all $B \subseteq A$, with $|B| = 3$, $L(B)_{\hat{\jmath}} \subseteq L(B)_k$, and

4.5.18.2 there are $x,y \in A$, $x \neq y$, such that

$L(\{x,y\}) = L(\{x,y\})_{\hat{\imath}} = L(\{x,y\})_{\hat{\jmath}}$.

Then (4.5.18.3) and (4.5.18.4) are equivalent.

4.5.18.3 There is a strongly Pareto-optimal, strongly non-dictatorial, and positively associated welfare function F from $L(A)_N$ to $W(A)$ on Γ.

4.5.18.4 There is a strongly Pareto-optimal, strongly non-dictatorial, and positively associated welfare function H from $L(A)_T$ to $W(A)$, on $\langle A,T \rangle$, where $|T| = 2$ and $T \subseteq N$.

Proof of theorem 4.5.18

(4.5.18.4) -> (4.5.18.3) is evident by (4.5.13)

(4.5.18.3) -> (4.5.18.4)

Without loss of generality suppose $F : L(A)_N \to L(A)$.

It is sufficient to prove that there exists a strongly Pareto-optimal, strongly non-dictatorial, and positively associated welfare function H from $L(A)_M$ to $L(A)$, on $\langle A,M \rangle$, such that $M \subset N$, and if $|M| \geq 3$, then $\hat{\imath},\hat{\jmath} \in M$.

Take $k \in N$. Without loss of generality let $L(B)_{\hat{\imath}} \subseteq L(B)_k$ for all $B \subseteq A$, $|B| = 3$.

Define F^k similarly to F^2 in the proof of (4.5.14).

Obviously $F^k : L(A)_{N_k} \to L(A)$ is strongly Pareto-optimal and positively associated.

If F^k is non-dictatorial we are done.

If F^k is dictatorial by (4.5.18.2) it is obvious that $\hat{\imath}$ is the dictator of F^k.

Hence, $D(H,\{\hat{\imath},k\}) = K(L(A)_N,\{\hat{\imath},k\})$.

Moreover, by (4.5.18.2), (4.5.5.1) follows.

Hence, by (4.5.5) we are done.

∎

Theorem 4.5.18 is related to theorems already known in literature, see e.g. Kalai & Muller [1977], and Ritz [1983].

There the conditions imposed on the welfare function are weaker, that is the independence of irrelevant alternatives is taken instead of the positively associativity, but the domains studied are more restrictive. Only domains of the following type are studied: for all $i,j \in N$ $L(A)_j = L(A)_i$. It is clear that the results are not related in a stronger or weaker relation, because of these incomparabilities. Although by their characterization it immediately follows by theorem 4.5.11 that positive associativity may be replaced by the independence contion.

By Ritz [1985] and theorem 4.5.18 the domains pointed out in (4.5.18) are characterized. Like the other charaterizing theorems, however, it is not easy to verify whether or not a given domain satisfies this characterization.

We end this monograph with some summarizing remarks. In the Theory of Social Choice collective group decisions are studied. Normally the formal models for these group decisions are based on three primitive notions, namely: individuals (the participants in the collective group decision), alternatives (the possible decisions) and orderings (the basis on which the collective decision). Compared to the first two primitive notions the latter notion is complex. Notwithstanding, there does not exist a model for orderings in literature.

In chapter 2 a classification system for orderings is formulated. Within this model sets of relations can be classified as sets of orderings. This classification is based on some constructive (binary) operations. If a set of relations is closed with respect to these operations, then it can be classified. It is shown that every well-known type of ordering can be classified. Furthermore, a generalization of the transitivity condition is introduced. Since a set of relations, satisfying this condition, can be classified as a set of orderings, this generalized transitivity condition is a very simple sufficient condition in order to classify such a set. Besides, minimal extensions are characterized, all classified sets of orderings between the linear and interval-orderings are determined, several sets of tournaments are classified and all order morphisms are found.

On this fundamental knowledge welfare functions and choice correspondences are studied in chapter 4. In §4.1 the notion of order morphism is extended to order morphism between sets of profiles. It turns out that Pareto-optimal, neutral and independent of irrelevant alternatives welfare functions uniquely correspond to order morphisms. This type of welfare function is often studied in Social Choice Theory. Furthermore, in §4.2 it is shown that specific choice correspondences, often studied in Social Choice Theory, also correspond uniquely to order morphism by means of a reconstruction principle.

By these fundamental results it is worthwhile to study order morphisms within the classification system. Let us call an order morphism to be simple if it is a projection on one of its coordinates (dictatorship). We are interested in non-simple order morphisms, since simple order morphisms correspond to simple (dictatorial) decision mechanisms and we think that such mechanisms are not the only practically known ones.

Particularly, we are interested in the relations between the domain and range of a non-simple order morphisms. For instance, taking for the domain the set of linear profiles and for the range the set of weak orderings, then by Arrow's paradox there are only simple order mophisms between those two sets.

To explain this relation between domain and range here, a new primitive notion of operationality is needed. This notion indicates a degree of complexity when working with a specific notion within a given set of other notions. Let $V := \{x \in R$: There is a minimal denotation of x which has at least 999 nines in it$\}$. Compared to the ε-relation V is not very operational, which becomes clear from the following questions. $\pi \in V$? $e \in V$? $\pi + e \in V$? On the other hand the empty set is operational with respect to the ε-relation.

In §4.3 we have shown that if this operationality of the range is in terms of transitivity, then only simple order morphisms are possible (See theorem 4.3.10 and 4.3.11). Note that cyclicity is a special type of transitivity (as introduced in chapter 2 definition 2.3.11). So operationality within transitivity conditions is a strong requirement because only simple order morphisms can satisfy this condition. Although this is not a very surprizing result, because there are many specific concrete parial results in literature connected to this one, here a pausible framework is formulated in which transitivity conditions on the range appear to be inherent to simple order morphisms (dictatorship).

Likewise, operationality, in terms of the constructive operations of the classification system, imposed on the range is inherent to simple order morphisms (See theorem 4.3.12). So for non-simple order morphisms the range cannot be constructed from the domain in finite steps within the classification system. Intuitively this means that collective orderings are much more less structured than individual orderings. For instance it is still an open question what the image of $L_3(U)$ is like under a non-simple order morphism (See (4.3.19) up to (4.3.23)).

Given these fundamental impossibility results, it is natural to weaken the conditions imposed on welfare functions. Especially the independence of irrelevant alternatives requierment seems suitable to be weakened. It requires that the collective preference can be deduced uniquely from pairwise comparisons of the alternatives. This is a strong requirement, but guarantees operationality of the welfare function. How should this condition be weakened such that not all of this operationality is lost?

In chapter 3 continuity properties for functions between discrete metric spaces have been defined. These properties are weaker than the independence condition for welfare functions (See theorem 4.4.15). In spite of this weakening only simple welfare functions remain to be possible (See theorems 4.4.26 up to 4.4.28). In the proofs of these theorems it becomes clear, even more than in the proofs of the impossibility theorems of §4.3, that the fact that the domain contains every possible profile is used. For instance maximal confilicts are used extensively (See lemma 4.4.20).

It is very unlikely that a collective decision rule should take every possible profile into account (unrestricted domains). Note that individual preferences are often interdependent or not totally arbitrary to choose. In §4.5 we therefore study restricted domains. In that section necessary and sufficient conditions, for several special types of domain restrictions, are found in order that there exists Pareto-optimal, positively associated and non-dictatorial welfare functions on such domains. Here the range is equal to the set of weak orderings. The necessary and sufficient conditions are technical and not all of these are easily verified in a given restricted domain. Therefore, it is possible to doubt this approach with respect to operationality. However, given the foregoing impossibilities and noting that this approach leads to possibility theorems it is possible to reverse the operationality argumentation and give it up to a certain degree. Namely by argueing that the problems studied in Social Choice Theory are complex and due to this lead to more complex model descriptions than one might think of at first sigh.

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SUBJECT REFERENCE AND SYMBOL REFERENCE TABLE

Sub_1, Sub_2	other substitution operators	(2.7.11),144
$x \sim y$: sucR	x and y succeed each other in R	(4.4.2),261
	symmetric	(2.2.8),(4.1.6),52,209
$\bar{s}R_A$	symmetric part of R_A	(2.2.3),(1.1.4),4,36
$\langle V,\tau\rangle$	topology	(3.2.1),152
τ_d	d-induced topology on $E_M \cup V$	(3.2.13),157
$\tau_{Mstandard}$	standard topology on metric space M	(3.2.15),158
T(U)	tournaments	88
$T_3(U)$, $T_k(U)$	special tournaments	(2,5,6),99
$T_{k,l}(U)$	special tournaments	103
$T_{k,l,m}(U)$	special tournaments	103
$\langle w_1,w_2\rangle$-transitive	transitivity	(2.3.8),(1.1.4),4,76
	classifiable transitivity	(2.3.11),80
$\bar{t}R_A$	transitive closure of R_A	(2.2.3),36
	negative transitivity	(2.2.8),52
	quasi - transitivity	(2.2.8),52
	P^t - transitivity	(2.2.8),52
	$P^tI^mP^t$ - transitivity	(2.2.8),52
	unanimity respecting	(4.2.2),217
$K(V_n(A),S)$	set of unanimity feasible pairs by S in $V_n(A)$	(1.6.1),26
U	inverse	34
U2	$:= \{\langle x,y\rangle \in U \times U : \lvert\{x,y\}\rvert = 2\}$	(4.3.2),230
	single valuedness	(1.5.4),25
	relative majority voting	(1.2.1),8
F	welfare function	(1.1.6),7
$\hat{F}$	complete welfare function	(4.1.5),208
w	word over an alfabet	(2.3.5),73
$w^{\bar{v}}$	conversed word	(2.3.5),73

For symbols concerning set operations see (1.1.3) page 3.

SAMENVATTING

In de Sociale Keuze Theorie bestudeert men collectieve groepsbeslissingen. Normaliter zijn de formele modellen voor die groepsbeslissingen gebaseerd op 3 primitieve begrippen tw: individuen (de collectieve beslissers), alternatieven (de mogelijke collectieve besluiten) en orderingen (de basis waarop 't collectief besluit tot stand verondersteld wordt te komen). Vergeleken met de andere twee primitieve begrippen is 't begrip ordering complex. Desondanks bestaat er geen model voor dit complex begrip.

In hoofdstuk 2 wordt er een classificatie voor orderingen gegeven. Binnen dit model kunnen verzamelingen van relaties als een verzameling van orderingen geclassificeerd worden. Deze classificatie geschiedt op basis van enkele constructieve bewerkingen. Als de verzameling van relaties gesloten is t.a.v. die bewerkingen, dan is hij classificeerbaar. Er wordt aangetoond dat alle bekende typen van orderingen classificeerbaar zijn. Ook wordt er een generalisatie van transitiviteits condities gegeven. Daar een verzameling van relaties, die aan deze generale eis voldoet, classificeerbaar is en daar de eis relatief eenvoudig is, hebben we een eenvoudig voldoende criterium. Bovendien worden minimale uitbreidingen gekarakteriseerd, alle classificeerbare verzamelingen tussen de lineaire ordenigen en de interval ordenigen beschreven, (met 't oog op orderingen) toernooien bestudeerd en de verzameling van alle orde morfismen afgeleid.

Vanuit deze basiskennis worden in hoofdstuk 4 voorkeurregels en keuzeregels bestudeerd. In §4.1 breiden we 't begrip orde morfisme voor ordeningen uit tot orde morfisme voor profiel verzamelingen. Het blijkt dat Pareto-optimale, neutrale en onafhankelijke van irrelevante alternatieven zijnde voorkeurregels een eenduidig overeenkomen met orde morfismen. Deze voorkeurregels worden vaak bestudeerd in de Sociale Keuze Theorie. Voorts blijkt in §4.2 dat vaak bestudeerde keuzeregels middels een reconstrueerbaarheids principe eveneens eeneenduidig corresponderen met orde morfismen.

De fundamentele resultaten van hierboven verantwoorden een studie in orde morfismen binnen 't classificeerbaarheids kader.

We noemen een orde morfismen eenvoudig als hij bijvoorbeeld een projectie is (dictatuur). Daar zo'n orde morfismen collectieve beslissingsmechanismen beschrijft, en deze laatste in de praktijk vaak als ingewikkeld ervaren worden en zeker niet uitsluitend als dictaturen, zijn we geïnteresseerd in de relatie tussen het domein en het bereik van een niet eenvoudig orde morfismen. Nemen we bijvoorbeeld als domein de verzameling van profielen bestaande uit lineaire orderingen en als bereik de verzameling van zwakke orderingen, dan betekent Arrow's paradox dat er uitsluitend eenvoudige (dictatoriale) orde morfismen tussen deze twee bestaan.

Om die relatie tussen domein en bereik te bespreken is een begrip als "hanteerbaarheid" nodig. Dit hier primitief veronderstelde begrip is een maat welke aangeeft hoe eenvoudig men met een notie binnen een stel andere gegeven noties om kan gaan. Bijvoorbeeld laat

V := {x ε R : Er is een decimale ontwikkeling van x, welke tenminste 999 verschillende keren 't cijfer 9 bevat}.

T.o.v. de ε - relatie is V hoogst waarschijnlijk niet zo hanteerbaar. Probeer de volgende vragen maar te beantwoorden: $\pi \in V$? $e \in V$? $\pi + e \in V$? De lege verzameling is t.a.v. deze ε-relatie hanteerbaar te noemen.

In §4.3 hebben we nu aangetoond dat als we het bereik binnen transitiviteits noties hanteerbaar willen houden, dan zijn er enkel eenvoudige orde morfismen mogelijk. (Zie stellingen 4.3.8, 4.3.10 en 4.3.11). Dit resultaat generalizeert vele stellingen uit de literatuur. Dus hanteerbaarheid binnen transitiviteits noties is een te sterke eis, daar hierbij alleen eenvoudige orde morfismen mogelijk zijn. Ofschoon dit geen schokkend resultaat is, gezien de vele in de literatuur gevonden deelresultaten, is hier echter wel een plausibel kader aangegeven waarbinnen blijkt dat transitiviteits condities voor het bereik onlosmakelijk verbonden zijn met de eenvoud van orde morfismen (dictaturen).

Evenzo is hanteerbaarheid van het bereik beschreven vanuit het domein, in termen van de constructieve bewerkingen, welke binnen het classificatie systeem voor handen zijn, onlosmakelijk verbonden met de eenvoud van orde morfismen (Zie stelling

4.3.12). Dus bij niet eenvoudige orde morfismen is het bereik niet met een eindig aantal stappen uit 't domein te construeren binnen 't classificatie systeem. Intuïtief betekent dit dat de collectieve orderingen veel minder gestructureerd zijn dan de individuele orderingen. Het is bijvoorbeeld een open vraag wat het beeld van $L_3(U)$ onder een orde morfisme F is, als F niet eenvoudig is (Zie (4.3.19) t/m (4.3.23)).

Gezien de fundamentele resultaten van hierboven ligt het nu voor de hand, dat we de eisen welke men aan een voorkeurregel oplegt afzwakt. Vooral de onafhankelijkheid van irrelevante alternatieven lijkt op het eerste gezicht een geschikte conditie om afgezwakt te worden. Zij stelt namelijk dat de gemeenschappelijke voorkeur eenduidig kan worden afgeleid uit paarsgewijs vergelijken van de alternatieven. Dit is een sterke eis en garandeert de hanteerbaarhied van de voorkeurregel. Hoe kan men deze conditie zinnig, binnen de context van hanteerbaarhied, afzwakken?

In hoofdstuk 3 worden continuïteits eigenschappen voor functies van en naar discrete metrische ruimten gedefinieerd, welke voor voorkeurregels zwakker zijn dan de onafhankelijheid van irrelevante alternatieven (Zie stelling 4.4.15). Het blijkt echter, ondanks deze afzwakking, dat enkel eenvoudige (dictatoriale) voorkeurregels mogelijk zijn (Zie stellingen 4.4.26 t/m 4.4.28). Voorts zien we bij de bewijzen van deze resultaten, meer nog dan bij de bewijzen in § 4.3, gebruikmaking van het feit dat alle mogelijke profielen voorhanden zijn in het domein. Onder andere alle maximale conflict profielen worden verondersteld in het domein te zitten (Zie lemma 4.4.20).

Het is zeer onwaarschijnlijk dat een collectieve beslissingsregel rekening dient te houden met alle mogelijke profielen, bedenk dat individuele voorkeuren onderling afhankelijk van elkaar kunnen zijn. In § 4.5 bestuderen we dan ook beperkte domeinen. In deze paragraaf worden noodzakelijke en voldoende eisen voor speciale typen van domeinbeperkingen gegeven, z.d.d. er niet dictatoriale Pareto-optimale en positief associatieve voorkeurregels op zo'n domein mogelijk zijn, waarbij het bereik binnen de zwakke ordeningen valt. Die noodzakelijke en voldoende eisen zijn vaak technisch en niet allen eenvoudig toestbaar voor een beperkt domein. Men is derhalve ook hier geneigd hun hanteerbaarheid in twijfel te trekken. Echter, gezien het voorafgaande, en in ogenschouw nemend dat het hier mogelijkheden betreft, kan men ook omgekeerd redeneren. Namelijk dat de problematiek, bestudeerd in de Sociale Keuze Theorie, complex is en dien ten gevolge tot meer complexe model beschrijvingen van deze problemen leiden zal, dan men in het begin verwacht.